

# Positive Harmonic Functions on Denjoy Domains in the Complex Plane

by

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## 1. Introduction and Statement of Main Results

We use the following standard terminology. We denote by Denjoy domain an open subset  $\Omega$  of the complex plane  $\mathbf{C}$  whose complement  $E := \overline{\mathbf{C}} \setminus \Omega$ , where  $\overline{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$ , is a subset of  $\overline{\mathbf{R}} := \mathbf{R} \cup \{\infty\}$ , where  $\mathbf{R}$  is the real axis (see [13]). Throughout the paper we rely on the following assumption. Each point of  $E$  (including the point at infinity) is regular for the Dirichlet problem in  $\Omega$ . Denote by  $\mathcal{P}_\infty = \mathcal{P}_\infty(\Omega)$  the cone of positive harmonic functions on  $\Omega$  which have vanishing boundary values at every point of  $E \setminus \{\infty\}$ . Independently, Ancona [4] and Benedicks [7] showed that either all functions in  $\mathcal{P}_\infty$  are proportional or  $\mathcal{P}_\infty$  is generated by two linearly independent (minimal) harmonic functions; that is, either  $\dim \mathcal{P}_\infty = 1$  or  $\dim \mathcal{P}_\infty = 2$  respectively. In other words, it means that the Martin boundary of  $\Omega$  has either one or two “infinite” points.

The results in [4] and [7] are proved for positive harmonic functions in domain  $\Omega \subset \mathbf{R}^n, n \geq 2$ . In this paper we focus on the case  $n = 2$  due to its extreme importance in the theory of entire functions, where positive harmonic functions and subharmonic functions in  $\mathbf{C}$  which are non-positive on a subset of the real line were the subject of research significantly earlier.

Bernstein [8] showed that if an entire function  $f$  satisfies

$$\limsup_{|z| \rightarrow \infty} \frac{\log |f(z)|}{|z|} =: \sigma_f < \infty \quad (1.1)$$

and

$$|f(x)| \leq 1 \quad (x \in \mathbf{R}),$$

then  $|f'(x)| \leq \sigma_f$  for any  $x \in \mathbf{R}$ .

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In some extensions of the Bernstein theorem (see, for example, [24], [2], [18], [27], [3], [19], [20] and [21]) the entire function  $f$  satisfies (1.1) as well as a new condition

$$|f(x)| \leq 1 \quad (x \in E),$$

where  $E \subset \mathbf{R}$  conforms to certain metric properties. Then the authors derived estimates on the growth of  $f$  in  $\mathbf{C}$  of the form

$$|f(z)| \leq (H_E(z))^{\sigma_f} \quad (z \in \mathbf{C}),$$

where  $H_E(z)$  is a “universal function” which does not depend on  $f$ .

We say that a subharmonic function  $u$  in  $\mathbf{C}$  has finite degree  $0 < \sigma < \infty$ , if

$$\limsup_{|z| \rightarrow \infty} \frac{u(z)}{|z|} = \sigma.$$

We denote by  $K_\sigma(E)$  the class of subharmonic in  $\mathbf{C}$  functions of finite degree no larger than  $\sigma$  and non-positive on  $E$ . Let

$$v(z) = v(z, K_\sigma(E)) := \sup\{u(z) : u \in K_\sigma(E)\} \quad (z \in \mathbf{C})$$

be the subharmonic majorant of the class  $K_\sigma(E)$ . It is known that  $v(z)$  is either finite everywhere on  $\mathbf{C}$  or equal to  $+\infty$  on  $\mathbf{C} \setminus E$ . The set  $E$  is said to be of type  $(\alpha)$  in the former case, and of type  $(\beta)$  in the latter.

**Theorem A.** ([19], [20, Theorem 3.3], [21, Remark 1])

- (a) Case  $(\alpha)$  holds if and only if  $\dim \mathcal{P}_\infty = 2$ ;
- (b) case  $(\beta)$  holds if and only if  $\dim \mathcal{P}_\infty = 1$ .

There is a close connection between the dimension of  $\mathcal{P}_\infty$  and the behavior of the Green function  $g_\Omega(\cdot, z)$  for  $\Omega$  with pole at  $z \in \Omega$  (see [32], [25] or [26] for further details on logarithmic potential theory). Let  $E^* := \mathbf{R} \setminus E$ .

**Theorem B.** ([15, Theorem 3], [16, Section VIII A.2]) *Let*

$$U(z) = U(z, E) := \int_{E^*} g_\Omega(t, z) dt \quad (z \in \Omega).$$

*Then*

- (a)  $\dim \mathcal{P}_\infty = 1$  if and only if  $U \equiv \infty$  in  $\Omega$ ;
- (b)  $\dim \mathcal{P}_\infty = 2$  if and only if  $U$  is finite everywhere on  $\Omega$ .

The problem of finding a geometric description of  $E$  such that  $\dim \mathcal{P}_\infty = 2$  or, equivalently, of  $E$  with the finite subharmonic majorants of classes  $K_\sigma(E)$

attracted attention of a number of researches (see [3], [7], [15], [28], [12], [29] and [30]).

One of the basic results in this area is the following Benedicks' criterion. Let

$$R(x, r) := \left\{ z \in \mathbf{C} : |\Re z - x| < \frac{r}{2}, |\Im z| < \frac{r}{2} \right\} \quad (x \in \mathbf{R}, r > 0).$$

For an arbitrary fixed  $\alpha$  with  $0 < \alpha < 1$  and every  $x \in \mathbf{R} \setminus \{0\}$ , let  $\beta_x(\cdot) = \beta_x(\cdot, \alpha, E)$  be the solution of the Dirichlet problem on  $R(x, \alpha|x|) \setminus E$  with boundary values  $\beta_x = 1$  on  $\partial R(x, \alpha|x|)$  and  $\beta_x = 0$  on  $E \cap R(x, \alpha|x|)$ .

**Theorem C.** ([7, Theorem 4]) *For every  $\alpha$  with  $0 < \alpha < 1$ ,*

(a)  *$\dim \mathcal{P}_\infty = 1$  if and only if*

$$\int_{|x| \geq 1} \frac{\beta_x(x) dx}{|x|} = \infty;$$

(b)  *$\dim \mathcal{P}_\infty = 2$  if and only if*

$$\int_{|x| \geq 1} \frac{\beta_x(x) dx}{|x|} < \infty.$$

Theorem C indicates that the dimension of  $\mathcal{P}_\infty$  only depends on the geometry of  $E$  near infinity.

Theorems 1 and 2 below provide a natural and intrinsic characterization of  $E$  with a given  $\dim \mathcal{P}_\infty$  in terms of the logarithmic capacity  $\text{cap}(S)$ ,  $S \subset \mathbf{C}$ , which appears most suitable for this theory. In these theorems we also connect the dimension of  $\mathcal{P}_\infty$  with continuous properties of the Green function  $g_\Omega$  in a neighborhood of infinity.

**Theorem 1** *The following conditions are equivalent:*

(i) *There exist points  $a_j, b_j \in E$ ,  $-\infty < j < \infty$  such that*

$$b_{j-1} \leq a_j < b_j \leq a_{j+1}, \quad \lim_{j \rightarrow \pm\infty} a_j = \pm\infty, \quad (1.2)$$

$$\bigcup_{j=-\infty}^{\infty} (a_j, b_j) \supset E^*, \quad (1.3)$$

$$\inf_{-\infty < j < \infty} \frac{\text{cap}(E \cap [a_j, b_j])}{\text{cap}([a_j, b_j])} > 0, \quad (1.4)$$

$$\sum_{j=-\infty}^{\infty} \left( \frac{b_j - a_j}{|a_j| + 1} \right)^2 < \infty; \quad (1.5)$$

(ii)  $\dim \mathcal{P}_\infty = 2$ ;

(iii)  $\limsup_{\Omega \ni t \rightarrow \infty} g_\Omega(t, z)|t| < \infty$  for any  $z \in \Omega$ .

For particular results of this kind, see [28, Theorem 2], [29, Theorem 8], [21, Theorem 4] and [11, Theorem 1.11].

Notice that if  $(a, \infty) \subset E^*$  or  $(-\infty, a) \subset E^*$  for some  $a \in \mathbf{R}$ , then, by Theorem C,  $\dim \mathcal{P}_\infty = 1$ .

**Theorem 2** *Let  $E \cap (a, \infty) \neq \emptyset$  and  $E \cap (-\infty, -a) \neq \emptyset$  for any  $a > 0$ . The following conditions are equivalent:*

(i) *There exist points  $\{a_j, b_j\}_{j=-N}^M$ , where  $M + N = \infty$ , such that  $a_j, b_j \in E$ ,*

$$b_{j-1} \leq a_j < b_j \leq a_{j+1},$$

$$\sup_j \frac{\text{cap}(E \cap [a_j, b_j])}{\text{cap}([a_j, b_j])} < 1, \quad (1.6)$$

$$\sum_{j=-N}^M \left( \frac{b_j - a_j}{|a_j + b_j| + 1} \right)^2 = \infty; \quad (1.7)$$

(ii)  $\dim \mathcal{P}_\infty = 1$ ;

(iii)  $\limsup_{\Omega \ni t \rightarrow \infty} g_\Omega(t, z)|t| = \infty$  for some  $z \in \Omega$ .

Next, we state some metric tests which immediately follow from the theorems above and which use the one-dimensional Lebesgue measure (length)  $|S|$  of a linear (Borel) set  $S \subset \mathbf{R}$ .

**Remark 1.** Since

$$\text{cap}([a_j, b_j]) = \frac{b_j - a_j}{4} \quad \text{and} \quad \text{cap}(E \cap [a_j, b_j]) \geq \frac{|E \cap [a_j, b_j]|}{4},$$

the existence of points  $a_j, b_j \in E$ ,  $-\infty < j < \infty$  satisfying (1.2), (1.3), (1.5) and the new inequality

$$\inf_{-\infty < j < \infty} \frac{|E \cap [a_j, b_j]|}{b_j - a_j} > 0, \quad (1.8)$$

is sufficient for the validity of the parts (ii) and (iii) of Theorem 1 (cf. [27, Lemma 1], [7, Theorem 5] and [20, Theorem 3.8]).

**Remark 2.** Let  $E^* = \cup_{j=-N}^M (c_j, d_j)$ , where  $M + N = \infty$ , be such that

$$d_{j-1} < c_j < d_j < c_{j+1}.$$

In Theorem 2, taking the system of points  $\{a_j, b_j\}$  to be the same as  $\{c_j, d_j\}$  we derive that the condition

$$\sum_{j=-N}^M \left( \frac{d_j - c_j}{|c_j + d_j| + 1} \right)^2 = \infty \quad (1.9)$$

implies each of the parts (ii) and (iii) of Theorem 2 (cf. [7, Theorem 5], [15, Theorem 4] and [20, Theorem 3.7]).

**Remark 3.** Let

$$\theta_E(t) := |E^* \cap [-t, t]| \quad (t > 0).$$

The condition

$$\int_1^\infty \frac{\theta_E^2(t) dt}{t^3} < \infty$$

yields each of the parts (ii) and (iii) of Theorem 1 (cf. [28, Theorem 4], [12, Theorem 1], [29, Theorem 13 and Theorem 17] and [31, Theorem 2.2 and Corollary 4.1]).

**Remark 4.** Let  $\theta(t), t \geq 1$  be any increasing function such that

$$0 < \theta(t) \leq 2t \quad (t \geq 1), \quad (1.10)$$

$$\int_1^\infty \frac{\theta^2(t) dt}{t^3} = \infty. \quad (1.11)$$

Then, there exists  $E$  such that

$$\theta_E(t) \leq \theta(t) \quad (t > 2), \quad (1.12)$$

and parts (ii) and (iii) of Theorem 2 hold (cf. [29, Section 4.4], [12, Theorem 2] and [31, Corollary 4.2]).

**Remark 5.** The continuous properties of the Green function  $g_\Omega$  at boundary points are of independent interest in potential theory (see, for example, [10], [22], [9], [5], [11], [31] and references therein). Using conformal invariance of the Green function and the linear transformation

$$w = \frac{d_0 - c_0}{2z - c_0 - d_0},$$

where  $(c_0, d_0)$  is any of the finite components of  $E^*$ , we can rephrase the part (i) $\Leftrightarrow$ (iii) of Theorem 1 in the following equivalent form (in this form we prove it in Section 5).

Let  $F \subset \mathbf{R}$  be a regular compact set with the complement  $G := \overline{\mathbf{C}} \setminus F$ , and let  $g_G(\cdot) = g_G(\cdot, \infty)$  be the Green function of  $G$  with pole at infinity. We assume

that  $F \subset [-1, 1] =: I$  and  $\pm 1, 0 \in F$ . Let  $F^* := \mathbf{R} \setminus F$ . The equivalence (i) $\Leftrightarrow$ (iii) in Theorem 1 can be restated as follows: *The following conditions are equivalent:*

(i') *There exist points  $a_j, b_j \in F$ ,  $-\infty < j < \infty$  such that*

$$-1 \leq a_{-1} < b_{-1} \leq a_{-2} < b_{-2} \leq \dots < 0 < \dots \leq a_1 < b_1 \leq a_0 < b_0 \leq 1,$$

$$\begin{aligned} \bigcup_{j=-\infty}^{\infty} (a_j, b_j) &\supset F^* \cap I, \\ \inf_{-\infty < j < \infty} \frac{\text{cap}(F \cap [a_j, b_j])}{\text{cap}([a_j, b_j])} &> 0, \end{aligned} \tag{1.13}$$

$$\sum_{j=-\infty}^{\infty} \left( \frac{b_j - a_j}{a_j} \right)^2 < \infty; \tag{1.14}$$

(iii')

$$\limsup_{G \ni z \rightarrow 0} \frac{g_G(z)}{|z|} < \infty. \tag{1.15}$$

The monotonicity of the Green function yields

$$g_G(z) \geq g_{\overline{\mathbf{C}} \setminus I}(z) \quad (z \in \mathbf{C} \setminus I),$$

that is, if  $F$  has the “highest density” at 0, then  $g_G$  has the “highest smoothness” at the origin. In particular,

$$g_G(iy) \geq g_{\overline{\mathbf{C}} \setminus I}(iy) > \frac{y}{2} \quad (0 < y < 1),$$

i.e.,

$$\limsup_{G \ni z \rightarrow 0} \frac{g_G(z)}{|z|} \geq \frac{1}{2} > 0.$$

In this regard, Remark 5 describes the metric properties of  $F$  such that  $g_G$  has the “highest smoothness” at 0 (see the recent remarkable result by Carleson and Totik [11, Theorem 1.11] for another description of sets  $F$  whose Green’s function possesses the property (1.15)).

The rest of the paper is organized as follows. In Section 2 we compile certain basic facts of the method by Akhiezer and Levin (see [3] and [20]), who connected the positive harmonic functions in  $\mathcal{P}_\infty$  with majorants of classes of subharmonic functions via special conformal mapping of the upper half-plane onto the upper half-plane with vertical slits (a comb domain). Since a significant number of the proofs in this paper depends on a new technique for estimation of the module of families of curves, Section 3 contains a brief summary of some special paths families and their modules. In Section 4 we continue to discuss the modules of

paths families (mainly crosscuts of a domain separating subsets and points of its closure). Section 5 is devoted to the proof of Theorem 1. In Section 6 we give the proof of Theorem 2. Remarks 3 and 4 are proved in Section 7.

## 2. Majorants in Classes of Subharmonic Functions

To describe the properties of subharmonic majorants of the class  $K_\sigma(E)$ , Akhiezer and Levin [3] introduced a special conformal mapping of the upper half-plane  $\mathbf{H} := \{z : \Im z > 0\}$  onto  $\mathbf{H}$  with vertical slits. Later, Levin [20] constructed the general theory of such mappings and used them to solve several extremal problems in classes of subharmonic functions. In this section we discuss some basic results of this theory which we use in the proofs of Theorem 1 and Theorem 2.

Recall that according to Theorem C we can assume that 0 belongs to  $E$  with some interval around the origin. Let  $E^* = \mathbf{R} \setminus E = \cup_{k=0}^M (c_k, d_k)$ , where  $c_k < d_k$ ,  $(c_k, d_k) \cap (c_j, d_j) = \emptyset$  if  $k \neq j$  and  $M \geq 0$  can be either finite or infinite. As before we assume that each point of  $E$  is regular for the Dirichlet problem in  $\Omega$  and that  $E \cap (a, \infty) \neq \emptyset$  and  $E \cap (-\infty, -a) \neq \emptyset$  for any  $a > 0$ . By [20, Theorem 3.1] there exists a conformal mapping  $\phi(\cdot) = \phi(\cdot, E)$  of  $\mathbf{H}$  onto

$$\mathbf{H}_E = \mathbf{H} \setminus \bigcup_{k=0}^M (u_k, u_k + iv_k],$$

where  $u_k \in \mathbf{R}$ ,  $u_k \neq u_j$  for  $k \neq j$  and  $v_k > 0$ , such that

$$\phi(0) := \lim_{\mathbf{H} \ni z \rightarrow 0} \phi(z) = 0, \quad \phi(\infty) := \lim_{\mathbf{H} \ni z \rightarrow \infty} \phi(z) = \infty,$$

$$\phi([c_k, d_k]) := \bigcup_{c_k \leq x \leq d_k} \lim_{\mathbf{H} \ni z \rightarrow x} \phi(z) = [u_k, u_k + iv_k],$$

and any point of  $[u_k, u_k + iv_k)$  has exactly two preimages. Besides,

$$\phi(E) := \bigcup_{x \in E} \lim_{\mathbf{H} \ni z \rightarrow x} \phi(z) = \overline{\mathbf{R}}.$$

Note that  $\phi$  is defined up to the multiplication by a positive constant. As a shorthand, we sometimes use  $\phi$  both for the conformal mapping of  $\mathbf{H}$  and for its continuous extension to  $\overline{\mathbf{H}}$ .

**Lemma 1** (*Theorem A and [3, Section n<sup>05</sup>]*).  $\dim \mathcal{P}_\infty = 2$  if and only if

$$\limsup_{y \rightarrow +\infty} \frac{|\phi(iy)|}{y} > 0.$$

Along with the set  $E$ , we consider the family of sets

$$E_R := E \cup \{x \in \mathbf{R} : |x| \geq R\} \quad (R > 0).$$

Let  $\phi_R(\cdot) := \phi(\cdot, E_R)$  be the appropriate conformal mapping. Multiplying by a positive constant if necessary, we normalize  $\phi_R$  such that

$$\lim_{\mathbf{H} \ni z \rightarrow \infty} \frac{\phi_R(z)}{z} = 1. \quad (2.1)$$

**Lemma 2** (*Theorem A, [3, Section n<sup>0</sup>3], and [21, Theorem 3]*).  $\dim \mathcal{P}_\infty = 2$  if and only if  $\Im \phi_R(z) = O(1)$  as  $R \rightarrow \infty$  at some  $z \in \mathbf{H}$ .

In the case of a regular (for the Dirichlet problem) compact set  $F \subset \mathbf{R}$  we use an analogue of a conformal mapping  $\phi$  to describe the Green function  $g_G(\cdot) = g_G(\cdot, \infty)$  for  $G = \overline{\mathbf{C}} \setminus F$  (with pole at  $\infty$ ) and the capacity of  $F$  (see [6] for details). Applying linear transformation if necessary we can always assume that  $F \subset [-1, 1] =: I$  and  $\pm 1 \in F$ . Consider the nontrivial case where  $F \neq I$ . The open (with respect to  $\mathbf{R}$ ) set  $F^* := \mathbf{R} \setminus F$  consists of either a finite number  $N > 2$  or infinite number  $N = \infty$  of open disjoint intervals, i.e.,

$$F^* = \bigcup_{j=1}^N (c_j, d_j),$$

where  $(c_1, d_1) := (-\infty, -1)$  and  $(c_2, d_2) := (1, \infty)$ . Let  $\mu_F$  be the equilibrium measure for the set  $F$ . Consider the function

$$\psi(z) = \psi(z, F) := \pi + i \left( \int_F \log(z - \zeta) d\mu_F(\zeta) - \log \text{cap}(F) \right) \quad (z \in \mathbf{H}).$$

It is analytic and univalent in  $\mathbf{H}$  and it maps  $\mathbf{H}$  onto a vertical half-strip with  $N - 2$  slits parallel to the imaginary axis, i.e., the domain

$$\Sigma_F = \{w : 0 < \Re w < \pi, \Im w > 0\} \setminus \bigcup_{j=3}^N [\tilde{u}_j, \tilde{u}_j + i\tilde{v}_j],$$

where  $0 < \tilde{u}_j < \pi$  and  $\tilde{v}_j > 0$ .

The continuous extension of  $\psi$  to  $\overline{\mathbf{H}}$  satisfies the following boundary correspondence

$$\begin{aligned} \psi(\infty) &= \infty, \quad \psi((-\infty, -1]) = \{w : \Re w = 0, \Im w \geq 0\}, \\ \psi([1, \infty)) &= \{w : \Re w = \pi, \Im w \geq 0\}, \quad \psi(F) = [0, \pi], \\ \psi([c_j, d_j]) &= [\tilde{u}_j, \tilde{u}_j + i\tilde{v}_j] \quad (j = 3, \dots, N). \end{aligned}$$



Note that in the last relation above, each point of  $[\tilde{u}_j, \tilde{u}_j + i\tilde{v}_j)$  has two preimages on  $[c_j, d_j]$ .

The advantage of using  $\psi$  lies in the fact that

$$g_G(z) = \Im\{\psi(z)\} \quad (z \in \overline{\mathbf{H}}).$$

Consider the function  $\Psi(z) := e^{i(\pi - \psi(z))}$ ,  $z \in \overline{\mathbf{H}}$ . Using the reflection principle we extend  $\Psi$  to a function analytic in  $\overline{\mathbf{C}} \setminus I$  according to the formula

$$\Psi(z) := \overline{\Psi(\bar{z})} \quad (z \in \mathbf{C} \setminus \overline{\mathbf{H}}).$$

Let  $\mathbf{D} := \{w : |w| < 1\}$  be the unit disk. The function  $\Psi$  is analytic and univalent and it maps  $\overline{\mathbf{C}} \setminus I$  onto a starlike (with respect to  $\infty$ ) domain  $\overline{\mathbf{C}} \setminus K_F$  with the following properties: it is symmetric with respect to the real line  $\mathbf{R}$  and it coincides with the exterior of the unit disk with  $2N - 4$  slits, i.e.,

$$\overline{\mathbf{H}} \cap K_F = (\overline{\mathbf{H}} \cap \overline{\mathbf{D}}) \cup \left( \bigcup_{j=3}^N [e^{i\theta_j}, r_j e^{i\theta_j}] \right),$$

where  $r_j := e^{\tilde{v}_j} > 1$  and  $0 < \theta_j := \pi - \tilde{u}_j < \pi$ .

There is a close connection between the capacities of the compact sets  $K_F$  and  $F$ , namely

$$\text{cap}(K_F) = \frac{1}{2 \text{cap}(F)} = \frac{\text{cap}(I)}{\text{cap}(F)}. \quad (2.2)$$

### 3. Modules of Path Families

The main idea of our approach is to estimate the (equal) modules of a family  $\Gamma$  of some paths in  $\mathbf{H}$  and the family  $\phi(\Gamma)$  of paths in  $\mathbf{H}_E$  or the family  $\psi(\Gamma)$  in  $\Sigma_F$  in various ways. We briefly recall the notion of the module of families of curves, generalized in an obvious way to path families, see [1], [17], [23] and [14] for details.

As usual, a Jordan curve is a continuous image of a closed interval without intersections (except possibly endpoints). By a curve we understand a locally rectifiable Jordan curve without endpoints. We define a path to be the union of finitely many mutually disjoint curves. For any path  $\gamma$ , denote by  $\bar{\gamma}$  the closure of  $\gamma$  in  $\mathbf{C}$ , i.e., the union of  $\gamma$  and the endpoints of curves composing  $\gamma$ .

A Borel measurable function  $\rho \geq 0$  on  $\mathbf{C}$  is a metric if

$$0 < A(\rho) := \int_{\mathbf{C}} \rho^2(z) dm(z) < \infty,$$

where  $dm(z)$  stands for the 2-dimensional Lebesgue measure (area) on  $\mathbf{C}$ .

For a path family  $\Gamma = \{\gamma\}$  let

$$L_\rho(\Gamma) := \inf_{\gamma \in \Gamma} \int_\gamma \rho(z) |dz|$$

(if the latter integral does not exist for some  $\gamma \in \Gamma$ , then we define it to be the infinity).

The quantity

$$m(\Gamma) := \inf_\rho \frac{A(\rho)}{L_\rho^2(\Gamma)}, \quad (3.1)$$

where the infimum is taken with respect to all metrics  $\rho$ , is called the module of the family  $\Gamma$ . In the sequel we refer to the basic properties of the module, such as conformal invariance, comparison principle, composition laws, etc. (see [1], [17], [23] and [14]). We will use these properties and the definition (3.1) without further citations.

Special families of separating paths play a rather useful role. Let  $D \subset \overline{\mathbf{C}}$  be a domain. We say that a path  $\gamma \in D$  separates sets  $A \subset \overline{D}$  and  $B \subset \overline{D}$  if any Jordan curve  $J \subset D$  joining  $A$  with  $B$  has nonempty intersection with  $\gamma$ .

We use  $\Gamma, \Gamma_1, \dots$  to denote path families. We may use the same symbol for different families if it does not lead to confusion.

The examples below state some well-known facts concerning special families of curves.

For  $w_0 \in \mathbf{R}$  and  $0 < r_1 < r_2$ , let

$$\Gamma_1 = \Gamma_1(w_0, r_1, r_2) := \{\gamma_r = \{w_0 + re^{i\theta} : 0 < \theta < \pi\} : r_1 < r < r_2\},$$

$$\Gamma_2 = \Gamma_2(w_0, r_1, r_2) := \{\gamma_\theta = \{w_0 + re^{i\theta} : r_1 < r < r_2\} : 0 < \theta < \pi\}.$$

Then

$$m(\Gamma_1) = \frac{1}{\pi} \log \frac{r_2}{r_1}, \quad (3.2)$$

$$m(\Gamma_2) = \frac{\pi}{\log \frac{r_2}{r_1}}. \quad (3.3)$$

Further, for  $a < b$  and  $c > 0$ , let  $\Gamma_3 = \Gamma_3(a, b, c)$  be the family of all half-ellipses in  $\mathbf{H}$  with the foci at  $a$  and  $b$  which have nonempty intersection with the interval  $(\frac{a+b}{2}, \frac{a+b}{2} + ic)$ . Applying the transformation

$$w \rightarrow \frac{b-a}{4} \left( w + \frac{1}{w} \right) + \frac{a+b}{2}$$

and (3.2) we obtain

$$\begin{aligned} m(\Gamma_3) &= \frac{1}{\pi} \log \left( \frac{2c}{b-a} + \sqrt{1 + \frac{4c^2}{(b-a)^2}} \right) \\ &> \frac{1}{2\pi} \log \left( 1 + \frac{4c^2}{(b-a)^2} \right). \end{aligned} \quad (3.4)$$

Let  $\Gamma_4 = \Gamma_4(r, R)$ ,  $0 < r < R$  be the family of all closed curves in  $\mathbf{H}$  which separate points  $ir$  and  $iR$  from  $\mathbf{R}$ . Applying the linear transformation

$$z \rightarrow \frac{z - ir}{z + ir},$$

we see that

$$m(\Gamma_4) = \frac{1}{2\pi} \mu \left( \frac{R-r}{R+r} \right) = \frac{\pi}{4\mu \left( \frac{r}{R} \right)},$$

where  $\mu(t)$ ,  $0 < t < 1$  is the module of Grötzsch's extremal domain  $\mathbf{D} \setminus [0, t]$  (see [17, pp. 53, 61]). Since

$$\mu(t) < \log \frac{4}{t} \quad (3.5)$$

(see [17, p. 61]), it follows that

$$m(\Gamma_4) \geq \frac{\pi}{4 \log \frac{4R}{r}}. \quad (3.6)$$

Further, denote by  $\Gamma_5 = \Gamma_5(r, R)$ ,  $0 < r < R$ , the family of all crosscuts of  $\mathbf{H}$ , i.e., curves in  $\mathbf{H}$  with endpoints on  $\mathbf{R}$ , which separate 0 and  $ir$  from  $iR$  and  $\infty$ . By symmetry

$$2m(\Gamma_4)2m(\Gamma_5) = 1.$$

Therefore, (3.6) implies that

$$m(\Gamma_5) = \frac{1}{4m(\Gamma_4)} \leq \frac{1}{\pi} \log \frac{4R}{r}.$$

For  $a < b < c$ , let  $\Gamma_6 = \Gamma_6(a, b, c)$  denote the family of all crosscuts of  $\mathbf{H}$  joining the boundary intervals  $(a, b)$  and  $(c, \infty)$ . The appropriate result for Teichmüller's extremal problem (see [17, p. 55]) and (3.5) yield

$$m(\Gamma_6) \leq \frac{1}{\pi} \log \frac{16(c-a)}{c-b}. \quad (3.7)$$

For  $a > 0$ , let  $\Gamma_7 = \Gamma_7(a)$  be the family of all curves joining the boundary intervals  $[0, ia]$  and  $\{z : \Re z = \pi, \Im z > 0\}$  in the half-strip  $\{z : 0 < \Re z < \pi, \Im z > 0\}$ . By [5, (4.8)-(4.9)] we have

$$m(\Gamma_7) \geq \begin{cases} \frac{1}{2}, & \text{if } a \geq \frac{\log 2}{2}, \\ \frac{\pi}{2 \log \frac{12}{a}} & \text{if } a \leq \frac{\log 2}{2}. \end{cases}$$

For  $a \in \mathbf{R}$  and  $0 < b < c$ , let  $\Gamma_8 = \Gamma_8(a, b, c)$  be the family of all curves  $\gamma \subset \Sigma = \Sigma(a, b, c) := \{z : a - \pi < \Re z < a + \pi, \Im z > b\}$  joining  $\{z : \Re z = a - \pi, \Im z > b\}$  and  $\{z : \Re z = a + \pi, \Im z > b\}$  in  $\Sigma$  such that  $\gamma \cap \{z : \Re z = a, \Im z > b\}$  consists of exactly one point which belongs to the interval  $(a + ib, a + ic)$ . The result of the previous example implies that

$$m(\Gamma_8) \geq \begin{cases} \frac{1}{4}, & \text{if } c - b \geq \frac{\log 2}{2}, \\ \frac{\pi}{4 \log \frac{12}{c-b}} & \text{if } c - b \leq \frac{\log 2}{2}. \end{cases} \quad (3.8)$$

For  $0 < a < b$  and  $c > 0$ , denote by  $\Gamma_9 = \Gamma_9(a, b, c)$  the family of all crosscuts of

$$G = G(a, b, c) = \mathbf{H} \setminus R_1,$$

where  $R_1 = R_1(a, b, c) := \{z : a \leq \Re z \leq b, 0 < \Im z \leq c\}$ , joining boundary intervals  $(0, a)$  and  $(b, \infty)$ . We claim that

$$m(\Gamma_9) \geq \frac{1}{2\pi} \log \frac{b^2}{c^2 + (b-a)^2}. \quad (3.9)$$

Indeed, in the nontrivial case where  $c^2 + (b-a)^2 < b^2$  we apply (3.2) to have

$$m(\Gamma_9) \geq m(\Gamma_1(b, \sqrt{c^2 + (b-a)^2}, b)) = \frac{1}{2\pi} \log \frac{b^2}{c^2 + (b-a)^2}.$$

Next, we discuss the following direct consequence of Pflüger's theorem (see [23, p. 212]). Let  $0 < a < b$  be such that  $\alpha := \log \frac{b}{a} < \frac{\pi}{2}$ , and let  $S \subset [a, b]$  be any set consisting of a finite number of closed intervals. Denote by  $\Gamma_{10} = \Gamma_{10}(a, b, A)$  the family of all paths in

$$R_2 = R_2(a, b) := \{re^{i\theta} : a < r < b, 0 < \theta < \alpha\}$$

which separate  $S$  from the interval  $(ae^{i\alpha}, be^{i\alpha}) := \{re^{i\theta} : a < r < b, \theta = \alpha\}$ . We claim that

$$m(\Gamma_{10}) \leq \frac{2}{\pi} \log \frac{C(b-a)}{\text{cap}(S)}, \quad C = \frac{e^\pi + 1}{2}. \quad (3.10)$$

To see this, consider the function

$$f(z) = \exp \left\{ i\pi \frac{\log \frac{z}{a}}{\log \frac{b}{a}} \right\}$$

which maps  $R_2$  conformally and univalently onto the half-ring

$$R_3 = \{\zeta \in \mathbf{H} : e^{-\pi} < |\zeta| < 1\}.$$

Since

$$|f(x_2) - f(x_1)| \geq \frac{2|x_2 - x_1|}{\alpha b} \quad (x_1, x_2 \in S),$$

for  $B := f(S)$ , we have

$$\text{cap}(B) \geq \frac{2}{\alpha b} \text{cap}(S). \quad (3.11)$$

Let  $B' := \{z \in \mathbf{C} \setminus \mathbf{H} : \bar{z} \in B\}$ . Pflüger's inequality (see [23, p. 212]) yields

$$\text{cap}(B) \leq \text{cap}(B \cup B') \leq (e^{\pi/2} + e^{-\pi/2}) \exp \left\{ -\frac{\pi}{m(\Gamma_{11})} \right\}, \quad (3.12)$$

where  $\Gamma_{11}$  is the family of all curves in the ring  $\{z : e^{-\pi} < |z| < 1\}$  joining  $B \cup B'$  with  $\{z : |z| = e^{-\pi}\}$ .

By symmetry and [12, Theorem IV.4.2]

$$m(\Gamma_{11}) = \frac{2}{m(\Gamma_{10})}.$$

Therefore, (3.11) and (3.12) imply (3.10).

Next, let

$$\begin{aligned} -R \leq \alpha_{-M} < u_{-M} < \beta_{-M} \leq \dots < \beta_{-2} \leq \alpha_{-1} < u_{-1} < \beta_{-1} \leq -r < 0 \\ < r \leq \alpha_0 < u_0 < \beta_0 \leq \alpha_1 < \dots \leq \alpha_N < u_N < \beta_N \leq R, \end{aligned}$$

where  $M > 0$  and  $N \geq 0$  are finite integers, and let real numbers  $u_j$  and  $v_j$ ,  $-M \leq j \leq N$  satisfy

$$0 \leq v_j \leq \min\{|u_j|, \beta_j - \alpha_j\}.$$

Denote by  $\Gamma_{12} = \Gamma_{12}(r, R, \{u_j\}, \{v_j\})$  the family of all curves in

$$D = D(r, R, \{u_j\}, \{v_j\}) := \{w \in \mathbf{H} : r < |w| < 2R\} \setminus \bigcup_{j=-M}^N [u_j, u_j + iv_j]$$

which join circular arcs  $\{z \in \mathbf{H} : |z| = r\}$  and  $\{z \in \mathbf{H} : |z| = 2R\}$ .

**Lemma 3** *The inequality*

$$m(\Gamma_{12}) \leq \frac{\pi \log \frac{2R}{r} - C \sum_{j=-M}^N \left( \frac{v_j}{u_j} \right)^2}{\left( \log \frac{2R}{r} \right)^2} \quad (3.13)$$

holds with the constant  $C = 10^{-5}$ .

*Proof.* Let  $\varepsilon := 10^{-1}$ , and for  $-M \leq j \leq N$ , let

$$B_j := \begin{cases} \{w \in D : |w - u_j| \leq \varepsilon^2 v_j, \Re w \leq u_j\}, & \text{if } u_j \geq \frac{1}{2}(\beta_j + \alpha_j), \\ \{w \in D : |w - u_j| \leq \varepsilon^2 v_j, \Re w \geq u_j\}, & \text{if } u_j < \frac{1}{2}(\beta_j + \alpha_j). \end{cases}$$

Consider the metric

$$\rho(w) = \begin{cases} |w|^{-1}, & \text{if } w \in D \setminus \cup_{j=-M}^N B_j, \\ 0, & \text{elsewhere.} \end{cases}$$

We proceed to show that

$$\int_{\gamma} \rho(w) |dw| \geq \log \frac{2R}{r} \quad (\gamma \in \Gamma_{12}). \quad (3.14)$$

Indeed, if  $\gamma \cap (\cup_{j=-M}^N B_j) = \emptyset$ , then

$$\int_{\gamma} \rho(w) |dw| \geq \left| \int_{\gamma} \frac{dw}{w} \right| \geq \log \frac{2R}{r}.$$

Moreover, (3.14) remains valid even if  $\gamma \cap (\cup_{j=-M}^N B_j) \neq \emptyset$ . We prove this as follows. Let  $j$  be such that  $\gamma \cap B_j \neq \emptyset$ . Denote by  $\gamma_j \subset \gamma$  the largest curve which has nonempty intersection with  $B_j$  and whose endpoints  $t_j$  and  $\tau_j$  are such that  $\varepsilon t_j, \varepsilon \tau_j \in \{w \in \partial B_j : |w - u_j| = \varepsilon^2 v_j\}$ . Let  $\gamma'_j \subset D \cap \{w : |w - u_j| = \varepsilon v_j\}$  be the circular arc with the same endpoints  $t_j$  and  $\tau_j$ . Since  $\gamma_j$  includes two curves joining  $\{w : |w - u_j| = \varepsilon^2 v_j\}$  and  $\{w : |w - u_j| = \varepsilon v_j\}$ , we conclude that

$$\int_{\gamma_j} \rho(w) |dw| \geq \frac{2\varepsilon(1 - \varepsilon)v_j}{|u_j| + \varepsilon v_j}.$$

Meanwhile, since  $\gamma'_j$  is a subarc of a circular part of  $\partial B_j$ , we obtain

$$\int_{\gamma'_j} \rho(w) |dw| \leq \frac{\pi}{2} \frac{\varepsilon v_j}{|u_j| - \varepsilon v_j}.$$

Hence, the choice of sufficiently small  $\varepsilon = 10^{-1}$  guaranties that

$$\int_{\gamma_j} \rho(w) |dw| \geq \int_{\gamma'_j} \rho(w) |dw|.$$

Therefore,

$$\int_{\gamma} \rho(w) |dw| \geq \int_{\gamma'} \rho(w) |dw| \geq \log \frac{2R}{r},$$

where

$$\gamma' = \left( \gamma \setminus \bigcup_{\gamma \cap B_j \neq \emptyset} \gamma_j \right) \cup \left( \bigcup_{\gamma \cap B_j \neq \emptyset} \gamma'_j \right) \subset D \setminus \bigcup_{j=-M}^N B_j.$$

This completes the proof of (3.14).

Since

$$A(\rho) \leq \pi \log \frac{2R}{r} - \frac{\pi}{4} \sum_{j=-M}^N \frac{\varepsilon^4 v_j^2}{(|u_j| + \varepsilon^2 v_j)^2},$$

according to (3.14) we have (3.13).

□

Further, let for  $T > 1$ ,

$$-T \leq a_{-M} < b_{-M} \leq a_{-M+1} < \dots \leq a_{-1} < b_{-1} \leq -1,$$

and

$$1 \leq a_0 < b_0 \leq a_1 < \dots \leq a_N < b_N \leq T,$$

where  $M > 0$  and  $N \geq 0$  are finite, be such that

$$\max_{-M \leq j \leq N} \frac{b_j - a_j}{\min\{|a_j|, |b_j|\}} < \frac{\pi}{2}.$$

Let

$$S^* \subset \bigcup_{j=-M}^N (a_j, b_j)$$

consist of a finite number of open (disjoint) intervals such that  $S := ([-T, -1] \cup [1, T]) \setminus S^*$  is a regular set satisfying

$$\min_{-M \leq j \leq N} \frac{\text{cap}(S \cap [a_j, b_j])}{\text{cap}([a_j, b_j])} \geq q > 0. \quad (3.15)$$

Denote by  $\Gamma_{13} = \Gamma_{13}(T, S, \{(a_j, b_j)\})$  the family of all paths in

$$Q = Q(T) := \{z \in \mathbf{H} : 1 < |z| < T\}$$

which separate either  $S \cap [-T, -1]$  from  $[1, T]$  or  $S \cap [1, T]$  from  $[-T, -1]$ .

**Lemma 4** *Under the above assumptions the inequality*

$$m(\Gamma_{13}) \leq \frac{\pi \log T + C \sum_{j=-M}^N \left( \frac{b_j - a_j}{\min\{|a_j|, |b_j|\}} \right)^2}{(\log T)^2}, \quad (3.16)$$

holds with the constant  $C = \frac{2}{\pi} \log \frac{10^2}{q}$ .

*Proof.* There is no loss of generality in assuming that  $S^* \cap (a_j, b_j) \neq \emptyset$  for any  $-M \leq j \leq N$ . Let

$$R_j := \begin{cases} \{z = re^{i\theta} : a_j \leq r \leq b_j, 0 \leq \theta \leq \log \frac{b_j}{a_j}\}, & \text{if } 0 \leq j \leq N, \\ \{z = re^{i\theta} : -b_j \leq r \leq -a_j, \pi - \log \frac{a_j}{b_j} \leq \theta \leq \pi\}, & \text{if } -M \leq j \leq -1. \end{cases}$$

Denote by  $\Gamma_j$  the family of all paths in  $R_j$  which separate  $S \cap [a_j, b_j]$  and the boundary interval

$$\begin{cases} \{z = re^{i\theta} : a_j \leq r \leq b_j, \theta = \log \frac{b_j}{a_j}\}, & \text{if } 0 \leq j \leq N, \\ \{z = re^{i\theta} : -b_j \leq r \leq -a_j, \theta = \pi - \log \frac{a_j}{b_j}\}, & \text{if } -M \leq j \leq -1. \end{cases}$$

According to (3.10) and our assumption (3.15) for  $j$  under consideration, there exists the metric  $\rho_j$  (with the support in  $R_j$ ) such that

$$\int_{\gamma} \rho_j(z) |dz| \geq 1 \quad (\gamma \in \Gamma_j)$$

and

$$A(\rho_j) \leq \frac{2}{\pi} \log \frac{(e^\pi + 1)(b_j - a_j)}{2\text{cap}(S \cap [a_j, b_j])} \leq \frac{2}{\pi} \log \frac{10^2}{q} =: C. \quad (3.17)$$

Consider the metrics

$$\rho^*(z) := \begin{cases} |z|^{-1}, & \text{if } z \in Q, \\ 0, & \text{elsewhere,} \end{cases}$$

$$\rho(z) := \max \left\{ \rho^*(z), \sum_{j=-M}^N \rho_j(z) \left| \log \frac{a_j}{b_j} \right| \right\}.$$

We claim that for any  $\gamma \in \Gamma_{13}$ ,

$$\int_{\gamma} \rho(z) |dz| \geq \log T. \quad (3.18)$$

Below we confirm the validity of (3.18) in the case where  $\bar{\gamma} \cap [-T, -1] = \emptyset$  (the proof of (3.18) in the case where  $\bar{\gamma} \cap [1, T] = \emptyset$  follows along the same lines). Let for  $\gamma \in \Gamma_{13}$  and  $1 \leq r_1 \leq r_2 \leq T$ ,

$$\gamma(r_1, r_2) := \{z \in \gamma : r_1 \leq |z| \leq r_2\}.$$

If  $\overline{\gamma(r_1, r_2)} \cap [r_1, r_2] = \emptyset$ , then

$$\int_{\gamma(r_1, r_2)} \rho(z) |dz| \geq \int_{\gamma(r_1, r_2)} \rho^*(z) |dz| \geq \left| \int_{\bar{\gamma}(r_1, r_2)} \frac{dz}{z} \right| \geq \log \frac{r_2}{r_1}, \quad (3.19)$$



where  $\tilde{\gamma}(r_1, r_2) \subset \gamma(r_1, r_2)$  is any curve joining  $\{z : |z| = r_1\}$  with  $\{z : |z| = r_2\}$ .

Furthermore, let  $\overline{\gamma(a_j, b_j)} \cap [a_j, b_j] \neq \emptyset$  for some  $0 \leq j \leq N$ . If there is no curve  $\tilde{\gamma}(a_j, b_j) \subset \gamma(a_j, b_j)$  joining the circles  $\{z : |z| = a_j\}$  with  $\{z : |z| = b_j\}$ , then at least one of the following two cases holds.

(a) If there exists  $\gamma'_j \subset \gamma(a_j, b_j)$  such that  $\gamma'_j \in \Gamma_j$ , then

$$\int_{\gamma(a_j, b_j)} \rho(z) |dz| \geq \left( \log \frac{b_j}{a_j} \right) \int_{\gamma'_j} \rho_j(z) |dz| \geq \log \frac{b_j}{a_j}. \quad (3.20)$$

(b) If there exists  $\gamma''_j \subset \gamma(a_j, b_j) \cap R_j$  joining  $\{z : \arg z = \log \frac{b_j}{a_j}\}$  with  $\mathbf{R}$ , then

$$\int_{\gamma(a_j, b_j)} \rho(z) |dz| \geq \int_{\gamma''_j} \rho^*(z) |dz| \geq \left| \int_{\gamma''_j} \frac{dz}{z} \right| \geq \log \frac{b_j}{a_j}. \quad (3.21)$$

Therefore, by (3.19)-(3.21) we obtain

$$\begin{aligned} \int_{\gamma} \rho(z) |dz| &\geq \sum_{j=0}^N \int_{\gamma(a_j, b_j)} \rho(z) |dz| + \sum_{j=0}^{N-1} \int_{\gamma(b_j, a_{j+1})} \rho(z) |dz| \\ &\quad + \int_{\gamma(1, a_0)} \rho(z) |dz| + \int_{\gamma(b_N, T)} \rho(z) |dz| \\ &\geq \sum_{j=0}^N \log \frac{b_j}{a_j} + \sum_{j=0}^{N-1} \log \frac{a_{j+1}}{b_j} + \log a_0 + \log \frac{T}{b_N} = \log T, \end{aligned}$$

which proves (3.18).

Since by (3.17)

$$\begin{aligned} A(\rho) &\leq A(\rho^*) + \sum_{j=-M}^N A(\rho_j) \left( \log \frac{b_j}{a_j} \right)^2 \\ &\leq \pi \log T + C \sum_{j=-M}^N \left( \frac{b_j - a_j}{\min\{|a_j|, |b_j|\}} \right)^2, \end{aligned}$$

by virtue of (3.18) and the definition of the module we have (3.16).

□

## 4. Preliminary Constructions

In this section we prove some auxiliary results.

Let  $F \subset I, F \neq I, \pm 1 \in F, F^*, \psi : \mathbf{C} \setminus I \rightarrow \Sigma_F$ , and  $\Psi : \overline{\mathbf{C}} \setminus I \rightarrow \overline{\mathbf{C}} \setminus K_F$  be defined as in Section 2, and let  $(c_j, d_j), 3 \leq j \leq N \leq \infty$  be the components of  $F^* \cap I$ . Denote by  $\Gamma_j$  the family of all paths  $\gamma \subset \mathbf{H}$  which separate  $F$  from  $\infty$  such that  $\bar{\gamma} \cap (c_j, d_j) \neq \emptyset$ .

Let  $j'$  be such that  $\tilde{v}_{j'} = \max_{3 \leq j \leq N} \{\tilde{v}_j\}$  and let  $r_{j'} := e^{\tilde{v}_{j'}}$ . By (2.2), (3.2) and the monotonicity of the capacity, we have

$$\begin{aligned} m(\Gamma_{j'}) &= m(\Psi(\Gamma_{j'})) \\ &\geq m(\{\gamma_r = \{w \in \mathbf{H} : |w| = r\} : 1 < r < r_{j'}\}) \\ &= \frac{1}{\pi} \log r_{j'} \geq \frac{1}{\pi} \log \text{cap}(K_F) = \frac{1}{\pi} \log \frac{\text{cap}(I)}{\text{cap}(F)}, \end{aligned}$$

i.e.,

$$\frac{\text{cap}(F)}{\text{cap}(I)} \geq \exp\{-\pi m(\Gamma_{j'})\}. \quad (4.1)$$

**Lemma 5** *Under the above assumptions and definitions the inequalities*

$$e^{-\tilde{v}_{j'}} \leq \frac{\text{cap}(F)}{\text{cap}(I)} \leq \frac{4}{2 + e^{\tilde{v}_{j'}} + e^{-\tilde{v}_{j'}}} \quad (4.2)$$

and

$$\tilde{v}_{j'} \geq \pi(m(\Gamma_j) - 1) \quad (4.3)$$

hold for any  $3 \leq j \leq N$ .

Furthermore, if  $\tilde{v}_{j'} \leq \frac{\pi}{4}$ , then for any  $3 \leq j \leq N$ ,

$$\tilde{v}_{j'} \geq \frac{\pi}{2} \exp\left(-\frac{10\pi}{m(\Gamma_j)}\right). \quad (4.4)$$

*Proof.* The monotonicity of the capacity yields

$$\begin{aligned} r_{j'} &= \text{cap}(\{w : |w| \leq r_{j'}\}) \geq \text{cap}(K_F) \\ &\geq \text{cap}(\{\overline{\mathbf{D}} \cup [1, r_{j'}]\}) = \frac{1}{4} \left(2 + r_{j'} + \frac{1}{r_{j'}}\right), \end{aligned}$$

which, together with (2.2), implies (4.2).

Next, we estimate from above the module of the family  $\Gamma'_j := \psi(\Gamma_j)$ . Consider the metric

$$\rho(w) := \begin{cases} 1, & \text{if } w \in \Sigma_F, \Im w \leq \tilde{v}_{j'} + \pi, \\ 0, & \text{elsewhere,} \end{cases}$$

Since for any  $\gamma \in \Gamma'_j$ ,

$$\int_{\gamma} \rho(w) |dw| \geq \pi,$$

we see that

$$m(\Gamma'_j) \leq \frac{A(\rho)}{\pi^2} = \frac{\tilde{v}_{j'} + \pi}{\pi},$$

from which (4.3) follows.

For the small values of  $\tilde{v}_{j'}$  we derive another estimate, i.e., inequality (4.4), which reflects the fact that  $m(\Gamma_j) = m(\Gamma'_j)$  can be arbitrary small. Let  $\tilde{v}_{j'} \leq \frac{\pi}{4}$ . Without loss of generality we can assume that  $\tilde{u}_j \geq \frac{\pi}{2}$  (if  $\pi$  is closer to  $\tilde{u}_j$  than 0 the reasoning below has to be modified in a straightforward way).

Consider the metric

$$\rho_j(w) := \begin{cases} |w - \tilde{u}_j|^{-1}, & \text{if } w \in \Sigma_F, \tilde{v}_{j'} \leq |w - \tilde{u}_j| \leq \pi, \\ 0, & \text{elsewhere,} \end{cases}$$

We claim that for any  $\gamma \in \Gamma'_j$ ,

$$\int_{\gamma} \rho_j(w) |dw| \geq \frac{1}{\sqrt{5}} \log \frac{\pi}{2\tilde{v}_{j'}}. \quad (4.5)$$

In order to prove (4.5), for  $\tilde{v}_{j'} \leq r < R \leq \tilde{u}_j$  we set

$$\begin{aligned} B_j(r, R) &:= \Sigma_F \bigcap \left( \{w = u + iv : r \leq |u - \tilde{u}_j| \leq R, 0 \leq v \leq \tilde{v}_{j'}\} \right. \\ &\quad \cup \left. \left\{ w = \tilde{u}_j + te^{i\theta} : r^2 + \tilde{v}_{j'}^2 \leq t^2 \leq R^2 + \tilde{v}_{j'}^2, \right. \right. \\ &\quad \left. \left. \sin^{-1} \left( \frac{\tilde{v}_{j'}}{t} \right) \leq \theta \leq \pi - \sin^{-1} \left( \frac{\tilde{v}_{j'}}{t} \right) \right\} \right). \end{aligned}$$

For  $\gamma \in \Gamma'_j$  and  $R \leq 2r$  we have

$$\int_{\gamma \cap B_j(r, R)} \rho_j(w) |dw| \geq \frac{R - r}{\sqrt{\tilde{v}_{j'}^2 + R^2}} \geq \frac{R - r}{\sqrt{5}r} \geq \frac{1}{\sqrt{5}} \log \frac{R}{r},$$

from which (4.5) immediately follows.

According to the definition of the module (3.1) and our assumption that  $\tilde{v}_{j'} \leq \frac{\pi}{4}$  we obtain

$$\begin{aligned} m(\Gamma'_j) &\leq 5 \left( \log \frac{\pi}{2\tilde{v}_{j'}} \right)^{-2} A(\rho_j) \\ &\leq 5 \pi \left( \log \frac{\pi}{2\tilde{v}_{j'}} \right)^{-2} \log \frac{\pi}{\tilde{v}_{j'}} \leq \frac{10\pi}{\log \frac{\pi}{2\tilde{v}_{j'}}}, \end{aligned}$$

which proves (4.4). □

We use Lemma 6 below in Section 6.

**Lemma 6** *Let  $a < b$  and let  $F \subset [a, b]$  be a regular compact set such that  $a, b \in F$  and*

$$\frac{\text{cap}(F)}{\text{cap}([a, b])} \leq e^{-3\pi}. \quad (4.6)$$

*Then, there exists a compact set  $F^*$  consisting of a finite number of closed intervals, such that  $F \subset F^* \subset [a, b]$  and*

$$0 < q' < \frac{\text{cap}(F^*)}{\text{cap}([a, b])} < q'' < 1 \quad (4.7)$$

*holds with absolute constants  $q'$  and  $q''$ .*

*Proof.* We can certainly assume that  $[a, b] = I$ . According to (4.6) and the left-hand side of (4.2) we have  $\tilde{v}_{j'} \geq 3\pi$ . Consider the set

$$F^* := \{x \in I : \Im \psi(x) \leq \tilde{v}_{j'} - 2\pi\}.$$

Let  $\psi^* : \mathbf{H} \rightarrow \Sigma_{F^*}, \{\tilde{u}_k^*\}$  and  $\{\tilde{v}_k^*\}$  be defined as in Section 2 for the compact set  $F^*$  instead of  $F$  and let

$$\tilde{v}_{k'}^* := \max_k \{\tilde{v}_k^*\}.$$

Denote by  $\Gamma_k^*$  the family of all paths  $\gamma \subset \mathbf{H}$  which separate  $F^*$  from  $\infty$  such that  $\bar{\gamma} \cap \psi^{*-1}((\tilde{u}_k^*, \tilde{u}_k^* + i\tilde{v}_k^*)) \neq \emptyset$ . Let  $k^*$  be defined such that

$$[\tilde{u}_{k^*}^*, \tilde{u}_{k^*}^* + i\tilde{v}_{k^*}^*] := \psi^* \circ \psi^{-1}([\tilde{u}_{j'} + i(\tilde{v}_{j'} - 2\pi), \tilde{u}_{j'} + i\tilde{v}_{j'}]).$$

Since

$$\begin{aligned} m(\Gamma_{k^*}^*) &= m(\psi(\Gamma_{k^*}^*)) \\ &\geq m(\{[it, \pi + it] : \tilde{v}_{j'} - 2\pi < t < \tilde{v}_{j'}\}) = 2, \end{aligned}$$

(4.3) written for  $F^\star$  instead of  $F$  shows that

$$\tilde{v}_{k'}^\star \geq \pi.$$

Therefore, the right-hand side of (4.2) implies the right-hand side of (4.7).

In order to prove the left-hand side of (4.7), we consider the metric

$$\rho(w) = \begin{cases} 1, & \text{if } 0 \leq \Re w \leq \pi, \tilde{v}_{j'} - 3\pi \leq \Im w \leq \tilde{v}_{j'} + \pi, \\ 0, & \text{elsewhere.} \end{cases}$$

Since

$$\begin{aligned} \frac{\tilde{v}_{k'}^\star}{\pi} &= m(\{[it, \pi + it] : 0 < t < \tilde{v}_{k'}^\star\}) \leq m(\psi^\star(\Gamma_{k'}^\star)) \\ &= m(\Gamma_{k'}^\star) = m(\psi(\Gamma_{k'}^\star)) \leq \frac{A(\rho)}{\pi^2} = 4, \end{aligned}$$

the left-hand side of (4.2) implies the left-hand side of (4.7).

□

Next, let  $E \subset \mathbf{R}$ ,  $\phi(\cdot) = \phi(\cdot, E)$ ,  $E^\star = \mathbf{R} \setminus E = \cup_j (c_j, d_j)$ , and  $[u_j, u_j + iv_j] = \phi([c_j, d_j])$  be defined as in Section 2. Consider  $\alpha, \beta \in \mathbf{R} \setminus (\cup_j u_j)$  such that  $\alpha < \beta$ . The points

$$a := \phi^{-1}(\alpha), \quad b := \phi^{-1}(\beta)$$

are uniquely defined. Let

$$v_{[\alpha, \beta]} := \sup\{v_j : \alpha < u_j < \beta\}.$$

**Lemma 7** (a) *If*

$$\frac{v_{[\alpha, \beta]}}{\beta - \alpha} \leq q_1, \tag{4.8}$$

*then there exists  $0 < q_2 = q_2(q_1) < 1$  such that*

$$\frac{\text{cap}(E \cap [a, b])}{\text{cap}([a, b])} \geq q_2. \tag{4.9}$$

(b) *If*

$$\frac{v_{[\alpha, \beta]}}{\beta - \alpha} \geq q_3 > 0, \tag{4.10}$$

*then there exists  $0 < q_4 = q_4(q_3) < 1$  such that*

$$\frac{\text{cap}(E \cap [a, b])}{\text{cap}([a, b])} \leq q_4. \tag{4.11}$$

(c) If

$$\frac{\text{cap}(E \cap [a, b])}{\text{cap}([a, b])} \leq q_5 < 1 \quad (4.12)$$

then there exists  $q_6 = q_6(q_5) > 0$  such that

$$\frac{v_{[\alpha, \beta]}}{\beta - \alpha} \geq q_6. \quad (4.13)$$

(d) If

$$\frac{\text{cap}(E \cap [a, b])}{\text{cap}([a, b])} \geq q_7 > 0 \quad (4.14)$$

then there exists  $q_8 = q_8(q_7) > 0$  such that

$$\frac{v_{[\alpha, \beta]}}{\beta - \alpha} \leq q_8. \quad (4.15)$$

*Proof.* Let  $F = h(E \cap [a, b])$ , where

$$h(z) = \frac{2z - (a + b)}{b - a}, \quad (4.16)$$

and let  $F^* \cap I = \cup_j (c_j, d_j)$ ,  $\psi : \mathbf{H} \rightarrow \Sigma_F$  and  $\{\tilde{u}_j\}$ ,  $\{\tilde{v}_j\}$  be defined as in Section 2 with changed, for the convenience, index numbering to be the same as the numbering of  $u_j$ 's and  $v_j$ 's, i.e.,

$$\psi \circ h \circ \phi^{-1}([u_j, u_j + iv_j]) = [\tilde{u}_j, \tilde{u}_j + i\tilde{v}_j].$$

Let  $j'$  be such that  $\tilde{v}_{j'} = \max_j \{\tilde{v}_j\}$  and let  $j^*$  be such that  $v_{j^*} = v_{[\alpha, \beta]}$ ,  $\alpha < u_{j^*} < \beta$ . As before, we denote by  $\Gamma_j$  the family of all paths  $\gamma \subset \mathbf{H}$  which separate  $F$  from  $\infty$  such that  $\bar{\gamma} \cap (c_j, d_j) \neq \emptyset$ . Let  $\tilde{\Gamma}_j$  be the family of all half-ellipses in  $\mathbf{H}$  with the foci at  $\alpha$  and  $\beta$  which have nonempty intersection with  $(u_j, u_j + iv_j]$ . By (3.4) we have

$$m(\Gamma_j) = m(\phi \circ h^{-1}(\Gamma_j)) \geq m(\tilde{\Gamma}_j) \geq \frac{1}{2\pi} \log \left( 1 + \frac{4v_j^2}{(\beta - \alpha)^2} \right). \quad (4.17)$$

Let  $\Gamma_j^*$  consist of all paths  $\gamma \subset \{w : \alpha < \Re w < \beta, \Im w > 0\}$  which separate  $(\alpha, \beta)$  from  $\infty$  such that  $\bar{\gamma} \cap (u_j, u_j + iv_j] \neq \emptyset$ . By the comparison principle we have

$$m(\Gamma_j) = m(\phi \circ h^{-1}(\Gamma_j)) \leq m(\Gamma_j^*). \quad (4.18)$$

Note also that

$$m(\Gamma_{j'}) = m(\psi(\Gamma_{j'})) \geq m(\{(it, \pi + it) : 0 < t < \tilde{v}_{j'}\}) = \frac{\tilde{v}_{j'}}{\pi}. \quad (4.19)$$

Next, applying the linear transformation  $w \rightarrow \frac{\pi(w-\alpha)}{\beta-\alpha}$  we state the analogues of (4.19), (4.3), and (4.4) for the module of the paths family  $\Gamma_j^*$ :

$$m(\Gamma_{j*}^*) \geq m(\{(\alpha + it, \beta + it) : 0 < t < v_{[\alpha, \beta]}\}) = \frac{v_{[\alpha, \beta]}}{\beta - \alpha}, \quad (4.20)$$

$$m(\Gamma_j^*) \leq \frac{v_{[\alpha, \beta]}}{\beta - \alpha} + 1, \quad (4.21)$$

and if  $\frac{v_{[\alpha, \beta]}}{\beta - \alpha} \leq \frac{1}{4}$  then

$$m(\Gamma_j^*) \leq \frac{10\pi}{\log \frac{\beta - \alpha}{2v_{[\alpha, \beta]}}}. \quad (4.22)$$

(a) By (4.8), (4.18) and (4.21)

$$m(\Gamma_{j'}) \leq q_1 + 1.$$

By (4.19)

$$\tilde{v}_{j'} \leq \pi(q_1 + 1).$$

Thus, applying the left-hand side of (4.2) and using the linear transformation (4.16) we obtain (4.9).

(b) By (4.10) and (4.17)

$$m(\Gamma_{j*}) \geq C_1 = C_1(q_3).$$

Since by (4.4)

$$\tilde{v}_{j'} \geq C_2 = C_2(C_1),$$

the right-hand side of (4.2) implies (4.11).

(c) By (4.12) and the left-hand side of (4.2)

$$\tilde{v}_{j'} \geq \log \frac{1}{q_5}.$$

Since by (4.18) and (4.19)

$$m(\Gamma_{j'}^*) \geq \frac{1}{\pi} \log \frac{1}{q_5},$$

(4.22) yields (4.13).

(d) By (4.14) and the right-hand side of (4.2)

$$\tilde{v}_{j'} \leq C_3 = C_3(q_7).$$

Since by (4.3)

$$m(\Gamma_{j*}) \leq \frac{C_3}{\pi} + 1,$$

(4.17) implies (4.15).

□

**Lemma 8** *Let  $\alpha\beta > 0$  and, consequently,  $ab > 0$ . Then*

$$\left(\frac{b-a}{b+a}\right)^2 \leq C \frac{(\beta-\alpha)^2 + v_{[\alpha,\beta]}^2}{(\beta+\alpha)^2} \quad (4.23)$$

*holds with the constant  $C = 2^{10}$ .*

*Proof.* Without loss of generality we assume that  $\beta > \alpha > 0$ . Let  $\Gamma = \Gamma(a, b, \mathbf{H})$  be the family of all crosscuts of  $\mathbf{H}$  which separate points  $a$  and  $b$  from points 0 and  $\infty$ . By (3.7)

$$m(\Gamma) \leq \frac{1}{\pi} \log \frac{16b}{b-a}. \quad (4.24)$$

Let  $\Gamma_1$  be the family of all crosscuts of  $\mathbf{H}_E$  which separate (in  $\mathbf{H}_E$ ) the rectangle

$$\{w : \alpha \leq \Re w \leq \beta, 0 \leq \Im w \leq v_{[\alpha,\beta]}\}$$

from 0 and  $\infty$ . By virtue of (3.9) we have

$$m(\Gamma_1) \geq m(\Gamma_9(\alpha, \beta, v_{[\alpha,\beta]})) \geq \frac{1}{2\pi} \log \frac{\beta^2}{(\beta-\alpha)^2 + v_{[\alpha,\beta]}^2}. \quad (4.25)$$

Since

$$m(\Gamma) = m(\phi(\Gamma)) \geq m(\Gamma_1),$$

comparing (4.24) and (4.25) with the last estimate we obtain (4.23):

$$\begin{aligned} \left(\frac{b-a}{b+a}\right)^2 &< \left(\frac{b-a}{b}\right)^2 \leq 16^2 \frac{(\beta-\alpha)^2 + v_{[\alpha,\beta]}^2}{\beta^2} \\ &\leq 2^{10} \frac{(\beta-\alpha)^2 + v_{[\alpha,\beta]}^2}{(\beta+\alpha)^2}. \end{aligned}$$

□

**Lemma 9** *Let  $E$  be such that*

$$\lim_{|u_j| \rightarrow \infty} \frac{v_j}{u_j} = 0. \quad (4.26)$$

*Then there exist points  $e_k, f_k, g_l, h_l \in E$  such that  $\phi(e_k), \phi(f_k), \phi(g_k), \phi(h_k) \in \mathbf{R} \setminus \cup_j u_j$ ,*

$$\begin{aligned} \dots &\leq e_{-1} < f_{-1} \leq 0 \leq e_0 < f_0 \leq e_1 < f_1 \leq \dots, \\ \dots &\leq g_{-1} < h_{-1} \leq 0 \leq g_0 < h_0 \leq g_1 < h_1 \leq \dots, \\ E^* &\subset [\cup_k (e_k, f_k)] \cup [\cup_l (g_l, h_l)], \quad (e_k, f_k) \cap (g_l, h_l) = \emptyset, \end{aligned}$$



$$\inf_k \frac{\text{cap}(E \cap [e_k, f_k])}{\text{cap}([e_k, f_k])} \geq q_1,$$

$$\sum_k \left( \frac{f_k - e_k}{|e_k| + 1} \right)^2 < \infty, \quad (4.27)$$

$$\begin{aligned} \frac{1}{6} &\leq \inf_l \frac{\max\{v_j : \phi(g_l) < u_j < \phi(h_l)\}}{\phi(h_l) - \phi(g_l)} \\ &\leq \sup_l \frac{\max\{v_j : \phi(g_l) < u_j < \phi(h_l)\}}{\phi(h_l) - \phi(g_l)} \leq \frac{1}{2}, \end{aligned} \quad (4.28)$$

$$q_2 \leq \inf_l \frac{\text{cap}(E \cap [g_l, h_l])}{\text{cap}([g_l, h_l])} \leq \sup_l \frac{\text{cap}(E \cap [g_l, h_l])}{\text{cap}([g_l, h_l])} \leq q_3, \quad (4.29)$$

where  $q_1 > 0$  and  $0 < q_2 < q_3 < 1$  are absolute constants.

*Proof.* Recall that by our assumption  $0 = \phi(0)$  belongs to  $E$  with a certain closed interval around the origin. First, we introduce points  $\alpha_m \in \mathbf{R} \setminus \cup_j u_j$  such that

$$\dots < \alpha_{-2} < \alpha_{-1} < \alpha_0 = 0 < \alpha_1 < \alpha_2 \dots, \quad 1 \leq \alpha_{m+1} - \alpha_m < 2.$$

Let

$$\delta_{k,m} := \max\{v_j : \alpha_k < u_j < \alpha_m\} \quad (k < m).$$

If  $\phi^{-1}([\alpha_k, \alpha_m]) \subset E$  we set  $\delta_{k,m} := 0$ . Next, we define a subsequence  $\{\gamma_s\}$  of a sequence  $\{\alpha_m\}$  in the following way.

Let  $\gamma_0 := \alpha_0 = 0$ . If  $\delta_{0,1} \leq \frac{1}{2}(\alpha_1 - \alpha_0)$  we set  $\gamma_1 := \alpha_1$ . Otherwise, i.e., if  $\delta_{0,1} > \frac{1}{2}(\alpha_1 - \alpha_0)$ , we consider the interval  $(\alpha_0, \alpha_2)$ . If

$$\delta_{0,2} \leq \frac{1}{2}(\alpha_2 - \alpha_0)$$

we set  $\gamma_1 := \alpha_2$ . Otherwise, i.e., if

$$\delta_{0,2} > \frac{1}{2}(\alpha_2 - \alpha_0),$$

we consider the interval  $(\alpha_0, \alpha_3)$ , etc. According to our assumption (4.26) after a finite number of steps we have

$$\delta_{0,m_1-1} > \frac{1}{2}(\alpha_{m_1-1} - \alpha_0) \geq \frac{1}{6}(\alpha_{m_1} - \alpha_0),$$

$$\delta_{0,m_1} \leq \frac{1}{2}(\alpha_{m_1} - \alpha_0).$$

We set  $\gamma_1 := \alpha_{m_1}$  and proceed in the same way to construct  $\gamma_2 := \alpha_{m_2}, m_2 > m_1$  such that either  $m_2 = m_1 + 1$  and

$$\delta_{m_1, m_2} \leq \frac{1}{2}(\alpha_{m_2} - \alpha_{m_1})$$

or  $m_2 > m_1 + 1$  and

$$\delta_{m_1, m_2-1} > \frac{1}{2}(\alpha_{m_2-1} - \alpha_{m_1}) \geq \frac{1}{6}(\alpha_{m_2} - \alpha_{m_1}),$$

$$\delta_{m_1, m_2} \leq \frac{1}{2}(\alpha_{m_2} - \alpha_{m_1}).$$

Repeating this procedure for both positive and negative indices  $m$  we obtain the sequence of real numbers

$$\dots < \gamma_{-2} < \gamma_{-1} < \gamma_0 = 0 < \gamma_1 < \gamma_2 < \dots$$

with the following properties.

(a) Either  $1 \leq \gamma_{s+1} - \gamma_s < 2$  and

$$\max\{v_j : \gamma_s < u_j < \gamma_{s+1}\} \leq \frac{1}{2}(\gamma_{s+1} - \gamma_s); \quad (4.30)$$

or

(b)

$$\frac{1}{6}(\gamma_{s+1} - \gamma_s) \leq \max\{v_j : \gamma_s < u_j < \gamma_{s+1}\} \leq \frac{1}{2}(\gamma_{s+1} - \gamma_s). \quad (4.31)$$

We denote by  $\mu_k$  and  $\mu_{k+1}$  in the case (a) and by  $\nu_l$  and  $\nu_{l+1}$  in the case (b) the endpoints of the intervals  $(\gamma_s, \gamma_{s+1})$ . Let

$$e_k := \phi^{-1}(\mu_k) \quad f_k := \phi^{-1}(\mu_{k+1}),$$

$$g_l := \phi^{-1}(\nu_l) \quad h_l := \phi^{-1}(\nu_{l+1}).$$

By (4.8)-(4.9) and (4.30) for any  $k$  under consideration, we have

$$\frac{\text{cap}(E \cap [e_k, f_k])}{\text{cap}([e_k, f_k])} \geq q_1 > 0.$$

Since by Lemma 8 and (4.30)

$$\left( \frac{f_k - e_k}{|e_k| + 1} \right)^2 \leq \frac{C}{k^2}$$

holds with some constant  $C > 0$ , we obtain (4.27).

Furthermore, (4.31) yields (4.28) which, together with parts (a) and (b) of Lemma 7, implies (4.29).

□

We state another result that can be proved in a similar manner.

**Lemma 10** *Let  $F \subset I, \pm 1 \in F$  be such that*

$$\lim_{\tilde{u}_j \rightarrow u_0} \frac{\tilde{v}_j}{\tilde{u}_j - u_0} = 0, \quad (4.32)$$

where  $u_0 := \psi(0)$ . Then there exist points  $\tilde{e}_k, \tilde{f}_k, \tilde{g}_l$ , and  $\tilde{h}_l \in F$  such that

$$-1 \leq \tilde{e}_{-1} < \tilde{f}_{-1} \leq \tilde{e}_{-2} < \dots < 0 < \dots < \tilde{f}_1 \leq \tilde{e}_0 < \tilde{f}_0 \leq 1,$$

$$-1 \leq \tilde{g}_{-1} < \tilde{h}_{-1} \leq \tilde{g}_{-2} < \dots < 0 < \dots < \tilde{h}_1 \leq \tilde{g}_0 < \tilde{h}_0 \leq 1,$$

$$F^* \cap I \subset \left[ \cup_k (\tilde{e}_k, \tilde{f}_k) \right] \cup \left[ \cup_l (\tilde{g}_l, \tilde{h}_l) \right], \quad (\tilde{e}_k, \tilde{f}_k) \cap (\tilde{g}_l, \tilde{h}_l) = \emptyset,$$

$$\inf_k \frac{\text{cap}(F \cap [\tilde{e}_k, \tilde{f}_k])}{\text{cap}([\tilde{e}_k, \tilde{f}_k])} \geq q_1,$$

$$\sum_k \left( \frac{\tilde{f}_k - \tilde{e}_k}{\tilde{e}_k} \right)^2 < \infty,$$

$$\begin{aligned} q_2 &\leq \inf_l \frac{\text{cap}(E \cap [\tilde{g}_l, \tilde{h}_l])}{\text{cap}([\tilde{g}_l, \tilde{h}_l])} \\ &\leq \sup_l \frac{\text{cap}(E \cap [\tilde{g}_l, \tilde{h}_l])}{\text{cap}([\tilde{g}_l, \tilde{h}_l])} \leq q_3, \end{aligned}$$

where  $q_1 > 0$  and  $0 < q_2 < q_3 < 1$  are absolute constants.

## 5. Proof of Theorem 1

*Proof of (i)  $\Rightarrow$  (ii).* Let  $E_R$  and  $\phi_R$  be defined as in Section 2. Denote by  $u_{j,R}$  and  $v_{j,R}$  the appropriate real numbers  $u_j$  and  $v_j$  defined in Section 2 for  $E_R$  instead of  $E$ . By Theorem C we can assume that  $[-e^{3\pi^2}, e^{3\pi^2}] \subset E$  and  $a_j \geq e^{2\pi^2}$  for  $j \geq 0$  as well as  $b_j \leq -e^{2\pi^2}$  for  $j < 0$ .

Since by (1.5)

$$\lim_{j \rightarrow \pm\infty} \frac{b_j - a_j}{a_j} = 0,$$

we can also assume that

$$\frac{b_j - a_j}{\min\{|a_j|, |b_j|\}} < \frac{\pi}{2}.$$

Let

$$K_t := \{z \in \mathbf{H} : |z| = t\} \quad (t > 0),$$

and let

$$u_R^+ := \inf\{u_{j,R} : u_{j,R} > 0\},$$

$$u_R^- := \sup\{u_{j,R} : u_{j,R} < 0\}.$$

We set  $u_R^+ := \infty$  if  $E^* \cap [0, R] = \emptyset$  and  $u_R^- := -\infty$  if  $E^* \cap [-R, 0] = \emptyset$ . Let

$$u_R := \min\{u_R^+, -u_R^-\} < \infty,$$

$$s_R := \sup\{|\phi_R(w)| : w \in K_1\}.$$

We start with the observation that the inequality

$$s_R \leq u_R \tag{5.1}$$

holds for sufficiently large  $R > e^{3\pi^2}$ .

Indeed, let  $z_R^* \in \overline{K_1}$  be such that for  $w_R^* := \phi_R(z_R^*)$  we have  $|w_R^*| = s_R$ . Assume, contrary to our claim (5.1), that

$$s_R = |w_R^*| > u_R.$$

Let  $\Gamma_1 = \{K_t : 1 < t < \exp(3\pi^2)\}$ . By (3.2) we have

$$m(\Gamma_1) = 3\pi.$$

Consider the family  $\Gamma'_1 = \Gamma'_1(R) := \phi_R(\Gamma_1)$  and the metric

$$\rho(w) := \begin{cases} 1, & \text{if } w \in \mathbf{H}, |w| \leq 2u_R, \\ 0, & \text{elsewhere.} \end{cases}$$

Since

$$m(\Gamma'_1) \leq \frac{A(\rho)}{u_R^2} = 2\pi,$$

we have the contradiction

$$3\pi = m(\Gamma_1) = m(\Gamma'_1) \leq 2\pi.$$

Hence, our assumption is incorrect, which proves (5.1).

In the reasoning below  $R$  and  $T > R$  are sufficiently large. By (2.1) we have

$$|\phi_R(z)| \leq 2T \quad (z \in K_T).$$

We proceed to show that

$$|\phi_R(z)| \geq C|w_R| \quad (z \in K_1), \quad (5.2)$$

where  $w_R := \phi_R(i)$  and  $C = \exp(-2\pi^2)$ .

To prove this, let  $w := \phi_R(z)$ ,  $z \in K_1$ . The only nontrivial case is where  $|w| < |w_R|$ . Let  $\Gamma'_2 = \Gamma'_2(w, w_R)$  be the family of all crosscuts of  $\mathbf{H}_{E_R}$  which separate 0 and  $w$  from  $w_R$  and  $\infty$ . Then (3.2) and (5.1) imply

$$m(\Gamma'_2) \geq m(\{K_t : |w| < t < |w_R|\}) = \frac{1}{\pi} \log \frac{|w_R|}{|w|}. \quad (5.3)$$

Considering the metric

$$\rho(z) := \begin{cases} 1, & \text{if } z \in \mathbf{H}, |z| \leq 2, \\ 0, & \text{elsewhere,} \end{cases}$$

for  $\Gamma_2 = \phi_R^{-1}(\Gamma'_2)$  we obtain

$$m(\Gamma_2) \leq A(\rho) = 2\pi. \quad (5.4)$$

Comparing (5.3) and (5.4) we have (5.2).

Let

$$t_R := \frac{1}{2}C|w_R|, \quad C = \exp(-2\pi^2).$$

According to (3.3), for the family of radial intervals

$$\Gamma'_3 = \Gamma'_3(R, T) = \{\gamma_\theta := \{it_R + re^{i\theta} : t_R < r < 2T\} : 0 < \theta < \pi\}$$

we have

$$m(\Gamma'_3) \geq \pi \left( \log \frac{2T}{t_R} \right)^{-1} = \pi \left( \log \frac{4T}{C|w_R|} \right)^{-1}. \quad (5.5)$$

Our next objective is to estimate the module of the path family  $\Gamma_3 = \phi_R^{-1}(\Gamma'_3)$  from above. Consider the set

$$\tilde{E}_R := \{x \in \mathbf{R} : \Im \phi_R(x) \leq t_R\} \supset E_R$$

which consists of a finite number of closed intervals.

Let  $\{(a_j, b_j)\}_{j=-J^-}^{J^+}$  be the intervals from the part (i) of Theorem 1 satisfying the condition

$$\mathbf{R} \setminus \tilde{E}_R =: \tilde{E}_R^* \subset \bigcup_{j=-J^-}^{J^+} (a_j, b_j) \subset (-T, T). \quad (5.6)$$

We assume that the system of intervals  $\{(a_j, b_j)\}_{j=-J^-}^{J^+}$  is minimal in the sense that  $J^\pm$  cannot be decreased with (5.6) still valid. It can happen that there are no such intervals at all (in this case we write  $J^- = 0$  and  $J^+ = -1$ ) or there are only intervals with positive endpoints (that is  $J^- = 0$ ) or there are only intervals with negative endpoints (that is  $J^+ = -1$ ).

Let  $\Gamma_{13} = \Gamma_{13}(T, \tilde{E}_R \cap ([1, T] \cup [-T, -1]), \{(a_j, b_j)\}_{j=-J^-}^{J^+})$  be the path family from Section 3. Notice that

$$m(\Gamma_3) \leq m(\Gamma_{13}) \quad (5.7)$$

and by (1.5) and Lemma 4

$$m(\Gamma_{13}) \leq \frac{\pi(\log T + C_1)}{(\log T)^2}. \quad (5.8)$$

Here and in the sequel we adopt the convention that  $C, C_1, C_2, \dots$  denote positive constants, possibly different in different cases.

Referring to (5.5), (5.7), and (5.8) we find that

$$(\log T)^2 \leq \left( \log \frac{4T}{C|w_R|} \right) (\log T + C_1),$$

and making  $T \rightarrow \infty$  we see that

$$\log |w_R| \leq \log \frac{4}{C} + C_1.$$

This means that  $|w_R| = |\phi_R(i)|$  is uniformly bounded and we can apply Lemma 2 to derive (ii).

□

*Proof of (ii)  $\Rightarrow$  (i).* We first show that (4.26) holds. Let  $\Gamma = \Gamma(y)$ ,  $y > 1$  be the family of all curves in  $\mathbf{H}$  which separate  $i$  and  $iy$  from  $\mathbf{R}$ . By (3.6) we have

$$m(\Gamma) \geq \frac{\pi}{4 \log 4y}. \quad (5.9)$$

Let  $\Gamma' := \phi(\Gamma)$ ,  $w_0 := \phi(i)$ , and  $w_1 := \phi(iy)$ . We assume that  $y$  is sufficiently large. In particular,  $y$  is so large that  $|w_0| < |w_1|$ . Let

$$Q = Q(|w_0|, |w_1|) := \{w \in \mathbf{H} : |w_0| \leq |w| \leq |w_1|\}.$$

Denote by  $\Gamma'_1$  the family of all curves in  $\mathbf{H}_E \cap Q$  joining circular arcs  $\{w : |w| = |w_0|\}$  and  $\{w : |w| = |w_1|\}$ . Since any  $\gamma \in \Gamma'$  includes two disjoint curves from the family  $\Gamma'_1$ , we conclude that

$$m(\Gamma') \leq \frac{m(\Gamma'_1)}{4}. \quad (5.10)$$

We prove (4.26) by contradiction. Suppose it were false. Then, we could find a constant  $0 < c < 1$  and a (monotone) sequence of integers  $\{j_k\}_{k=-M}^N$ , where  $M + N = \infty$ , such that

$$\begin{aligned} v_{j_k} &\geq c|u_{j_k}|, \\ u_{j_{k+1}} &> 2u_{j_k} > 2|w_0| \quad (j_k \geq 0), \\ u_{j_{k-1}} &< 2u_{j_k} < -2|w_0| \quad (j_k < 0). \end{aligned}$$

Consider the domain

$$D = D(\{u_{j_k}\}, c, |w_1|) := Q \setminus \bigcup_{k=-M_1}^{N_1} [u_{j_k}, u_{j_k}(1 + ic)],$$

where  $M_1 \leq M$  and  $N_1 \leq N$  are such that for  $0 \leq k \leq N_1$  or  $-M_1 \leq k \leq -1$  we have

$$|u_{j_k}| < \frac{|w_1|}{2}.$$

Notice that

$$m(\Gamma'_1) \leq m(\Gamma'_2), \quad (5.11)$$

where  $\Gamma'_2$  is the family of all curves in  $D$  joining circular arcs  $\{w : |w| = |w_0|\}$  and  $\{w : |w| = |w_1|\}$ .

Applying Lemma 3 with  $r = |w_0|$  and  $R = \frac{|w_1|}{2}$  we obtain

$$m(\Gamma'_2) \leq \frac{\pi \log \frac{|w_1|}{|w_0|} - C(M_1 + N_1)}{\left(\log \frac{|w_1|}{|w_0|}\right)^2}. \quad (5.12)$$

According to Lemma 1 we can choose  $y$  to be arbitrarily large and to satisfy the inequality

$$y \leq C_1 |w_1|. \quad (5.13)$$

Therefore, comparing (5.9)-(5.12) for such values of  $y$  we have

$$\frac{\pi}{\log 4C_1 |w_1|} \leq \frac{\pi \log C_2 |w_1| - C(M_1 + N_1)}{(\log C_2 |w_1|)^2},$$

i.e.,

$$M_1 + N_1 \leq \frac{C_3 \log C_2 |w_1|}{\log C_1 |w_1|}.$$

Passing to the limit as  $y \rightarrow \infty$ , i.e., as  $|w_1| \rightarrow \infty$  we have  $M + N \leq C_3$ . This contradiction proves (4.26).

As a cover  $\{(a_j, b_j)\}$  of  $E^*$  satisfying (1.2)-(1.5) we use the cover  $\{(e_k, f_k)\}$ ,  $\{(g_l, h_l)\}$ , constructed for any  $E$  satisfying (4.26) in Lemma 9. Thus, we only need to show that

$$\sum_l \left( \frac{h_l - g_l}{|g_l| + 1} \right)^2 < \infty. \quad (5.14)$$

Let  $\alpha_l := \phi(g_l)$ ,  $\beta_l = \phi(h_l)$ . According to (4.28) we have

$$\frac{1}{6}(\beta_l - \alpha_l) \leq v'_l := \max\{v_j : \alpha_l < u_j < \beta_l\} \leq \frac{1}{2}(\beta_l - \alpha_l). \quad (5.15)$$

Let  $u'_l \in (\alpha_l, \beta_l)$  be such that  $[u'_l, u'_l + iv'_l] \subset \partial \mathbf{H}_E$  and let  $l^\pm \leq L^\pm$  be defined such that for  $l^+ \leq l \leq L^+$  and  $-L^- \leq l \leq -l^-$ ,

$$(\alpha_l, \beta_l) \subset \left[ |w_0|, \frac{|w_1|}{2} \right] \cup \left[ -\frac{|w_1|}{2}, -|w_0| \right]$$

and  $v'_l < |u'_l|$ .

Consider the domain

$$D_1 = D_1(\{u'_l\}, \{v'_l\}, |w_0|, |w_1|) := Q \setminus \bigcup_{\substack{l^+ \leq l \leq L^+ \\ -L^- \leq l \leq -l^-}} [u'_l, u'_l + iv'_l].$$

Notice that

$$m(\Gamma'_1) \leq m(\Gamma'_3), \quad (5.16)$$

where  $\Gamma'_3$  is the family of all curves in  $D_1$  joining circular arcs  $\{w : |w| = |w_0|\}$  and  $\{w : |w| = |w_1|\}$ .

By Lemma 3 we have

$$m(\Gamma'_3) \leq \frac{\pi \log \frac{|w_1|}{|w_0|} - C_4 \sum_{\substack{l^+ \leq l \leq L^+ \\ -L^- \leq l \leq -l^-}} \left( \frac{v'_l}{u'_l} \right)^2}{\left( \log \frac{|w_1|}{|w_0|} \right)^2}. \quad (5.17)$$



Comparing (5.9), (5.10), (5.16), and (5.17) for  $y$  satisfying (5.13) we have

$$\frac{1}{\log 4C_1|w_1|} \leq (\log C_2|w_1|)^{-2} \left( \log C_2|w_1| - \frac{C_4}{\pi} \sum_{\substack{l^+ \leq l \leq L^+ \\ -L^- \leq l \leq -l^-}} \left( \frac{v'_l}{u'_l} \right)^2 \right),$$

i.e.,

$$\sum_{\substack{l^+ \leq l \leq L^+ \\ -L^- \leq l \leq -l^-}} \left( \frac{v'_l}{u'_l} \right)^2 \leq C_5 \frac{\log C_2|w_1|}{\log 4C_1|w_1|}.$$

Passing to the limit as  $y \rightarrow \infty$  and applying (5.15) we obtain

$$\sum_l \frac{(\beta_l - \alpha_l)^2 + v_l'^2}{(\beta_l + \alpha_l)^2} \leq C_6 \sum_l \left( \frac{v'_l}{u'_l} \right)^2 \leq C_7.$$

The last inequality and Lemma 8 imply (5.14). □

As we mentioned in the introduction, we can reformulate the part (i)  $\Leftrightarrow$  (iii) of Theorem 1 in the form of the equivalence (i')  $\Leftrightarrow$  (iii') of Remark 5.

*Proof of (i')  $\Rightarrow$  (iii') of Remark 5.* We use the same idea as in the proof of the part (i)  $\Rightarrow$  (ii) of Theorem 1.

Let  $u_0 := \psi(0)$ , and let for  $r > 0$ ,

$$K_r := \{z \in \mathbf{H} : |z| = r\}, \quad K'_r := \psi(K_r).$$

Denote by  $w_r \in K'_r$  any point with the property  $\Re w_r = u_0$  and let  $z_r := \psi^{-1}(w_r)$ .

Note that for any  $w \in K'_r$ ,

$$|w - u_0| \leq C|w_r - u_0|, \quad C = \exp(2\pi^2). \quad (5.18)$$

Indeed, if  $|w - u_0| > |w_r - u_0|$ , we consider the family  $\Gamma = \Gamma(r)$  of all crosscuts of  $\mathbf{H}$  which separate 0 and  $z_r$  from  $z := \psi^{-1}(w)$  and  $\infty$ . Considering the metric

$$\rho(\zeta) = \begin{cases} 1, & \text{if } \zeta \in \mathbf{H}, |\zeta| \leq 2r, \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$m(\Gamma) \leq \frac{A(\rho)}{r^2} = 2\pi. \quad (5.19)$$

On the other hand, by (3.2) for the family  $\Gamma' = \psi(\Gamma)$  we obtain

$$m(\Gamma') \geq m(\{K_t : |w_r - u_0| < t < |w - u_0|\}) = \frac{1}{\pi} \log \frac{|w - u_0|}{|w_r - u_0|}. \quad (5.20)$$

Comparing (5.19) and (5.20) we have (5.18).

According to (5.18), in order to verify (1.15), it is enough to show that

$$|w_r - u_0| \leq C_1 r \quad (5.21)$$

holds with a constant  $C_1 > 0$  independent of  $r$  for sufficiently small values of  $r > 0$ . In particular, we can assume that  $|w_r - u_0| < \frac{\log 2}{2}$ .

Consider the family  $\Gamma'_1 = \Gamma_8(a, b, c)$  from Section 3 with  $a = u_0, b = \frac{|w_r - u_0|}{3}$  and  $c = \Im w_r = |w_r - u_0|$ . By (3.8)

$$m(\Gamma'_1) \geq \frac{\pi}{4 \log \frac{18}{|w_r - u_0|}}. \quad (5.22)$$

Let

$$\tilde{F}_r := \left\{ x \in I : \Im \psi(x) \leq \frac{|w_r - u_0|}{3} \right\} \supset F.$$

Note that  $\tilde{F}_r$  consists of a finite number of closed intervals.

Denote by  $\Gamma_2 = \Gamma_2(r)$  the family of all paths in a half-ring  $\{z \in \mathbf{H} : r < |z| < 1\}$  which separate either  $\tilde{F}_r \cap [r, 1]$  from  $[-1, -r]$  or  $\tilde{F}_r \cap [-1, -r]$  from  $[r, 1]$ . The conformal invariance of the module and the comparison principle imply

$$m(\Gamma'_1) \leq \frac{1}{4} m(\psi(\Gamma_2)) = \frac{1}{4} m(\Gamma_2). \quad (5.23)$$

Let  $\Gamma_3$  consist of all  $\gamma \in \Gamma_3$  possessing the following property: there exists  $\gamma^* \in \Gamma_2$  such that  $\gamma = \{z : \frac{1}{z} \in \gamma^*\}$ .

Applying Lemma 4 and the assumptions (1.13)-(1.14) in the same way as we did in the proof of (5.8) we see that

$$m(\Gamma_2) = m(\Gamma_3) \leq \pi \left( \log \frac{C_2}{r} \right)^{-2} \log \frac{C_3}{r}. \quad (5.24)$$

Comparing (5.22)-(5.24) we obtain (5.21).

□

*Proof of (iii')  $\Rightarrow$  (i') of Remark 5.* We use essentially the same idea as in the proof of the part (ii)  $\Rightarrow$  (i) of Theorem 1. Let  $\tilde{u}_j$  and  $\tilde{v}_j$  be defined as in Section 2. By [11, Corollary 1.12] and [6, Theorem 1] we have

$$\lim_{t \rightarrow 0^+} \frac{\text{cap}(F \cap [0, t])}{\text{cap}([0, t])} = \lim_{t \rightarrow 0^+} \frac{\text{cap}(F \cap [-t, 0])}{\text{cap}([-t, 0])} = 1,$$

which implies (4.32).

To see this, let for  $\alpha \in (u_0, \pi) \setminus \cup_j \tilde{u}_j$ ,

$$\delta(\alpha) := \max\{\tilde{v}_j : u_0 < \tilde{u}_j < \alpha\},$$

and let  $j^*$  be such that  $u_0 < \tilde{u}_{j^*} < \alpha$  and  $v_{j^*} = \delta(\alpha)$ . Let  $t := \psi^{-1}(\alpha)$ . Denote by  $\Gamma_\alpha$  the family of all paths  $\gamma \subset \mathbf{H}$  which separate  $F \cap [0, t]$  from  $\infty$  such that  $\bar{\gamma} \cap \psi^{-1}((u_{j^*}, u_{j^*} + iv_{j^*})) \neq \emptyset$ . By Lemma 5, applied to the set  $\{x \in I : \frac{t}{2}(x+1) \in F \cap [0, t]\}$  instead of  $F$ , we obtain

$$\frac{\text{cap}(F \cap [0, t])}{\text{cap}([0, t])} \leq \frac{4}{2 + e^{v(t)} + e^{-v(t)}},$$

where

$$v(t) \geq \begin{cases} \pi(m(\Gamma_\alpha) - 1), & \text{if } v(t) > \frac{\pi}{4}, \\ \frac{\pi}{2} \exp\left(-\frac{10\pi}{m(\Gamma_\alpha)}\right), & \text{if } v(t) \leq \frac{\pi}{4}. \end{cases}$$

Passing to the limit as  $\alpha \rightarrow u_0^+$ , i.e., as  $t \rightarrow 0^+$ , we have

$$\lim_{t \rightarrow 0^+} v(t) = 0,$$

that is,

$$\lim_{\alpha \rightarrow u_0^+} m(\Gamma_\alpha) = 0. \quad (5.25)$$

Let  $\Gamma'_1 = \Gamma'_1(t)$  be the family of all half-ellipses in  $\mathbf{H}$  with the foci at  $u_0$  and  $\alpha$  which have nonempty intersection with the vertical interval  $(\tilde{u}_{j^*}, \tilde{u}_{j^*} + i\tilde{v}_{j^*})$ . Since

$$m(\Gamma_\alpha) = m(\psi(\Gamma_\alpha)) \geq m(\Gamma'_1), \quad (5.26)$$

according to (3.4), (5.25) and (5.26) we obtain

$$\lim_{\alpha \rightarrow u_0^+} \frac{\delta(\alpha)}{\alpha - u_0} = 0. \quad (5.27)$$

The same conclusion can be drawn for

$$\delta(\alpha) := \max\{\tilde{v}_j : \alpha < \tilde{u}_j < u_0\} \quad 0 < \alpha < u_0,$$

i.e.,

$$\lim_{\alpha \rightarrow u_0^-} \frac{\delta(\alpha)}{u_0 - \alpha} = 0. \quad (5.28)$$

The relations (5.27) and (5.28) imply (4.32).

As a cover  $\{(a_j, b_j)\}$  of  $F^* \cap I$  satisfying (1.13) and (1.14) we can use the cover  $\{(\tilde{e}_k, \tilde{f}_k)\}, \{(\tilde{g}_l, \tilde{h}_l)\}$ , for any  $F$  satisfying (4.32) discussed in Lemma 10. In this case we only need to show that

$$\sum_l \left( \frac{\tilde{h}_l - \tilde{g}_l}{\tilde{g}_l} \right)^2 < \infty. \quad (5.29)$$

Let  $\Gamma_2 = \Gamma_2(y), 0 < y < 1$  be the family of all curves in  $\mathbf{H}$  which separate  $i$  and  $iy$  from  $\mathbf{R}$ . By (3.6) we have

$$m(\Gamma_2) \geq \frac{\pi}{4 \log \frac{4}{y}}. \quad (5.30)$$

Let  $\Gamma'_2 := \psi(\Gamma_2)$ ,  $w_0 := \psi(i)$ , and  $w_1 := \psi(iy)$ . We assume that  $y$  is sufficiently small. In particular, it is such that  $2|w_1 - u_0| < d_0 := \min\{u_0, \pi - u_0, |w_0 - u_0|\}$ . Let

$$Q := \{w \in \mathbf{H} : |w_1 - u_0| \leq |w - u_0| \leq d_0\}$$

and let  $\tilde{\alpha}_l := \psi(\tilde{g}_l), \tilde{\beta}_l = \psi(\tilde{h}_l)$ . Furthermore, let  $\tilde{u}'_l \in (\tilde{\alpha}_l, \tilde{\beta}_l)$  be such that  $[\tilde{u}'_l, \tilde{u}'_l + i\tilde{v}'_l] \subset \partial\Sigma_F$ , and let positive integers  $l^\pm \leq L^\pm$  be defined so that for  $l^+ \leq l \leq L^+$  and  $-L^- \leq l \leq -l^-$ ,

$$(\tilde{\alpha}_l, \tilde{\beta}_l) \subset \left[ u_0 + |w_1 - u_0|, u_0 + \frac{d_0}{2} \right] \cup \left[ u_0 - \frac{d_0}{2}, u_0 - |w_1 - u_0| \right]$$

and  $\tilde{v}_l \leq |\tilde{u}_l - u_0|$ .

Consider the domain

$$D = D(\{\tilde{u}'_l\}, \{\tilde{v}'_l\}, |w_1 - u_0|, d_0) := Q \setminus \bigcup_{\substack{l^+ \leq l \leq L^+ \\ -L^- \leq l \leq -l^-}} [\tilde{u}'_l, \tilde{u}'_l + i\tilde{v}'_l].$$

Notice that

$$m(\Gamma'_2) \leq \frac{1}{4} m(\Gamma'_3), \quad (5.31)$$

where  $\Gamma'_3$  is the family of all curves in  $D$  joining circular arcs  $\{w : |w - u_0| = |w_1 - u_0|\}$  and  $\{w : |w - u_0| = d_0\}$ .

By Lemma 3 we have

$$m(\Gamma'_3) \leq \frac{\pi \log \frac{d_0}{|w_1 - u_0|} - C \sum_{\substack{l^+ \leq l \leq L^+ \\ -L^- \leq l \leq -l^-}} \left( \frac{\tilde{v}'_l}{\tilde{u}'_l - u_0} \right)^2}{\left( \log \frac{d_0}{|w_1 - u_0|} \right)^2}. \quad (5.32)$$

Further, proceeding as in the proof of the part (ii)  $\rightarrow$  (i) of Theorem 1 we derive the inequality (5.29) from (5.30)-(5.32). Since the proof of (5.29) is similar, we leave out the details of the proof.

□

## 6. Proof of Theorem 2

The equivalence (ii) $\Leftrightarrow$ (iii) is trivial (see Theorem C and Theorem 1).

*Proof of (i) $\Rightarrow$ (ii).* Let

$$\alpha_j := \phi(a_j), \beta_j := \phi(b_j), \delta_j := \sup\{v_k : \alpha_j < u_k < \beta_j\}.$$

If necessary slightly moving points  $a_j$  and  $b_j$ , we can assume that  $\alpha_j, \beta_j \in \mathbf{R} \setminus \cup_{j=-N}^M u_j$ . We also assume that  $\lim_{j \rightarrow \infty} \frac{\delta_j}{\beta_j} = 0$  if  $M = \infty$  and that  $\lim_{j \rightarrow -\infty} \frac{\delta_j}{\alpha_j} = 0$  if  $N = \infty$  (because otherwise (4.26) does not hold and by the reasoning in the beginning of the proof of the part (ii) $\Rightarrow$ (i) of Theorem 1 we have  $\dim \mathcal{P}_\infty \neq 2$ , i.e., by Theorem C  $\dim \mathcal{P}_\infty = 1$ ).

Furthermore, extending  $E \cap [a_j, b_j]$  for each interval  $[a_j, b_j]$  with the property

$$\frac{\text{cap}(E \cap [a_j, b_j])}{\text{cap}([a_j, b_j])} < e^{-3\pi}$$

in the way described in Lemma 6, we obtain a new set  $E^* \supset E$  which satisfies (1.6) in the strengthened form

$$0 < q' \leq \inf_j \frac{\text{cap}(E^* \cap [a_j, b_j])}{\text{cap}([a_j, b_j])} \leq \sup_j \frac{\text{cap}(E^* \cap [a_j, b_j])}{\text{cap}([a_j, b_j])} \leq q'' < 1, \quad (6.1)$$

where  $q'$  and  $q''$  are constants.

Since  $\dim \mathcal{P}_\infty(\mathbf{C} \setminus E^*) \geq \dim \mathcal{P}_\infty(\mathbf{C} \setminus E)$ , it is enough to show that  $\dim \mathcal{P}_\infty(\mathbf{C} \setminus E^*) = 1$ . Hence, we can proceed assuming that  $E = E^*$ .

The parts (c) and (d) of Lemma 7 and (6.1) imply

$$C_1(\beta_j - \alpha_j) \leq \delta_j \leq C_2(\beta_j - \alpha_j). \quad (6.2)$$

Since by (1.7) and (4.23) we have

$$\sum_{j=-N}^M \frac{(\beta_j - \alpha_j)^2 + \delta_j^2}{(\beta_j + \alpha_j)^2} = \infty,$$

(6.2) yields

$$\sum_{j=-N}^M \left( \frac{\delta_j}{\alpha_j} \right)^2 = \infty. \quad (6.3)$$

We claim that

$$\lim_{y \rightarrow +\infty} \frac{|\phi(iy)|}{y} = 0. \quad (6.4)$$

Indeed, as in the proof of the part (ii) $\Rightarrow$ (i) of Theorem 1 we let  $w_0 := \phi(i)$  and  $w_1 := \phi(iy)$ ,  $y > 1$ . We are interested in the case of sufficiently large values of  $y$ . Let  $\Gamma = \Gamma(y)$  be the family of all curves in  $\mathbf{H}$  which separate  $i$  and  $iy$  from  $\mathbf{R}$ . By (5.9), (5.10), (5.16), and (5.17) we have

$$\frac{\pi}{4 \log 4y} \leq m(\Gamma) \leq \frac{\pi \log \frac{|w_1|}{|w_0|} - C_3 \sum_{\substack{l^+ \leq l \leq L^+ \\ -L^- \leq l \leq -l^-}} \left( \frac{\delta_l}{\alpha_l} \right)^2}{4 \left( \log \frac{|w_1|}{|w_0|} \right)^2},$$

i.e.,

$$\sum_{\substack{l^+ \leq l \leq L^+ \\ -L^- \leq l \leq -l^-}} \left( \frac{\delta_l}{\alpha_l} \right)^2 \leq \frac{\pi}{C_3} \frac{\log \frac{|w_1|}{|w_0|}}{\log 4y} \log \frac{4|w_0|y}{|w_1|}.$$

Comparing the last inequality with (6.3), we obtain (6.4) which, together with Lemma 1, implies  $\dim \mathcal{P}_\infty = 1$ .

□

*Proof of (ii) $\Rightarrow$ (i).* First, let (4.26) hold. Consider the cover  $\{(e_k, f_k)\}$  and  $\{(g_l, h_l)\}$  of  $E^*$  from Lemma 9. We can take the system of intervals  $(g_l, h_l)$  to be our intervals  $(a_j, b_j)$ . By virtue of (4.29) they satisfy (1.6). They also satisfy (1.7) because otherwise the whole system of intervals  $\{(e_k, f_k)\}$  and  $\{(g_l, h_l)\}$  satisfies (1.5), which would mean by Theorem 1 that  $\dim \mathcal{P}_\infty = 2$ , a contradiction.

We now proceed with the case where

$$\limsup_{|u_j| \rightarrow \infty} \frac{v_j}{|u_j|} > 0.$$

Let  $u_{j_k}, v_{j_k}$  and  $0 < c < 1$  be such that

$$\lim_{k \rightarrow \infty} |u_{j_k}| = \infty, \quad v_{j_k} \geq c|u_{j_k}|, \quad v_{j_{k+1}} \geq v_{j_k}.$$

There is no loss of generality in assuming that  $u_{j_k} > 0$  and  $u_{j_{k+1}} > 4u_{j_k}$  (the case of the infinite number of negative  $u_{j_k}$ 's can be treated similarly).

Let  $\gamma_k^\pm \in \mathbf{R} \setminus \cup_j u_j$  satisfy

$$\gamma_k^- < u_{j_k} < \gamma_k^+, \quad \gamma_k^+ - \gamma_k^- = u_{j_k},$$

and let  $a_k := \phi^{-1}(\gamma_k^-)$ ,  $b_k := \phi^{-1}(\gamma_k^+)$ .

Our next objective is to show that the system of intervals  $\{(a_k, b_k)\}$  constructed above satisfies (1.6) and (1.7).

By (4.10)-(4.11) with  $a = a_k$  and  $b = b_k$  there exists  $0 < q < 1$  such that

$$\frac{\text{cap}(E \cap [a_k, b_k])}{\text{cap}([a_k, b_k])} < q,$$

from which (1.6) follows.

Let  $\Gamma_k$  be the family of all crosscuts of  $\mathbf{H}$  which separate points  $a_k$  and  $b_k$  from 0 and  $\infty$ . By (3.2) for its module we have

$$\begin{aligned} m(\Gamma_k) &\geq m(\{\gamma_r := \{z \in \mathbf{H} : |z - b_k| = r\} : b_k - a_k < r < b_k\}) \\ &= \frac{1}{\pi} \log \frac{b_k}{b_k - a_k} \cdot v_{j_k} \end{aligned} \quad (6.5)$$

Let  $\Gamma'_k = \phi(\Gamma_k)$ . Consider the metric

$$\rho(w) := \begin{cases} 1, & \text{if } -v_{j_k} \leq \Re w \leq u_{j_k} + v_{j_k}, 0 \leq \Im w \leq 2v_{j_k}, \\ 0, & \text{elsewhere.} \end{cases}$$

Since

$$\int_{\gamma} \rho(w) |dw| \geq v_{j_k} \quad (\gamma \in \Gamma'_k),$$

it follows that

$$m(\Gamma'_k) \leq \frac{A(\rho)}{v_{j_k}^2} = \frac{2(2v_{j_k} + u_{j_k})}{v_{j_k}} \leq 2 \left( 2 + \frac{1}{c} \right) =: C. \quad (6.6)$$

Comparing (6.5) and (6.6) we obtain

$$\frac{b_k - a_k}{b_k} \geq e^{-\pi C},$$

from which (1.7) follows.

□

## 7. Other Proofs

*Proof of Remark 3.* Our objective is to construct a cover of  $E^*$  satisfying the conditions of Remark 1.

First, we divide  $\mathbf{R}$  by points  $\pm 2^k$ ,  $k = 0, 1, \dots$  on the infinite number of intervals of the form either  $I_k^+ = [2^k, 2^{k+1}]$ , or  $I_k^- = [-2^{k+1}, -2^k]$ , or  $I_0 = I = [-1, 1]$  (here  $k = 1, 2, \dots$ ).

For each interval  $I' = I_k^+$  with the property

$$|I' \cap E^*| \geq 2^{k-1}, \quad (7.1)$$

we have

$$\int_{2^k}^{2^{k+1}} \frac{\theta_E^2(t)}{t^3} dt \geq \int_{3 \cdot 2^{k-1}}^{2^{k+1}} \frac{(t - 3 \cdot 2^{k-1})^2}{t^3} dt = \log \frac{4}{3} - \frac{9}{32} > 10^{-3}.$$

The same inequality is valid for  $I'$  of the form  $I' = I_k^-$  satisfying (7.1). Therefore, there is only a finite number of intervals, satisfying (7.1). According to Theorem C we can assume that  $E^* \subset \cup_{k=k_0}^{\infty} I_k^{\pm}$  for sufficiently large  $k_0$  such that

$$|I_k^{\pm} \cap E^*| < \frac{1}{2} |I_k^{\pm}| \quad (k \geq k_0).$$

For any interval of the covering of  $E^*$  (say  $I_k^+$ ) which has nonempty intersection with  $E^*$ , the linear transformation of the result of the lemma in [5, p. 580] (with its obvious extension to the case of sets consisting of the infinite number of intervals) implies the existence of real numbers

$$2^k \leq c_{k,1}^+ < d_{k,1}^+ \leq c_{k,2}^+ < \dots \leq c_{k,n_k^+}^+ < d_{k,n_k^+}^+ \leq 2^{k+1}$$

such that

$$I_k^+ \cap E^* \subset \bigcup_{j=1}^{n_k^+} [c_{k,j}^+, d_{k,j}^+],$$

$$|[c_{k,j}^+, d_{k,j}^+] \cap E^*| \geq \frac{e_{k,j}^+}{4}, \quad e_{k,j}^+ := d_{k,j}^+ - c_{k,j}^+.$$

Notice that

$$\begin{aligned} \sum_{j=1}^{n_k^+} \left( \frac{d_{k,j}^+ - c_{k,j}^+}{|c_{k,j}^+| + 1} \right)^2 &\leq 2^{-2k} \sum_{j=1}^{n_k^+} e_{k,j}^{+2} \leq 2^{-2k} \left( \sum_{j=1}^{n_k^+} e_{k,j}^+ \right)^2 \\ &\leq 2^{4-2k} \theta_E^2(2^{k+1}) \leq 2^8 \int_{2^{k+1}}^{2^{k+2}} \frac{\theta_E^2(t)}{t^3} dt. \end{aligned}$$

Consider the system of intervals  $\{(c_{k,j}^{\pm}, d_{k,j}^{\pm})\}_{j=1}^{n_k^{\pm}}\}_{k \geq k_0}$  constructed as above for each interval  $I_k^{\pm}, k \geq k_0$  which has nonempty intersection with  $E^*$ . To use Remark 1 we need to cover  $E^*$  by an infinite number of intervals going in both positive and negative directions of  $\mathbf{R}$ . To satisfy this formal condition we can add to the intervals constructed above the intervals of the form  $(n, n+1) \subset E$  or  $(-n -$



$1, -n) \subset E$ , where  $n$  is a positive integer if there is only a finite number of  $I_k^\pm$  such that  $I_k^\pm \cap E^* \neq \emptyset$ .

As a result we obtain the cover of  $E^*$  satisfying (1.2), (1.3), (1.5), and (1.8).

□

*Proof of Remark 4.* Following the idea from [31, Section 3.1] we set  $E = \mathbf{R} \setminus E^*$ , where

$$E^* = \bigcup_{j=1}^{\infty} \left( 2^j, 2^j + \frac{1}{4} (\theta(2^j) - \theta(2^{j-1})) \right) =: \bigcup_{j=1}^{\infty} (c_j, d_j).$$

Since for  $j \geq 1$  and  $2^j < t \leq 2^{j+1}$ ,

$$\theta_E(t) \leq \sum_{k=1}^j \frac{1}{4} (\theta(2^k) - \theta(2^{k-1})) = \frac{1}{4} \theta(2^j) \leq \theta(t),$$

we obtain (1.12).

Setting  $s_j := 2^{-j} \theta(2^j)$ , for any  $n > 1$  we have

$$\begin{aligned} \sum_{j=1}^n \left( \frac{d_j - c_j}{|c_j| + 1} \right)^2 &\geq 2^{-6} \sum_{j=1}^n \left( \frac{\theta(2^j) - \theta(2^{j-1})}{2^j} \right)^2 = 2^{-6} \sum_{j=1}^n \left( s_j - \frac{1}{2} s_{j-1} \right)^2 \\ &= 2^{-6} \left( \sum_{j=1}^n s_j^2 - \sum_{j=1}^n s_j s_{j-1} + \frac{1}{4} \sum_{j=1}^n s_{j-1}^2 \right) \\ &\geq 2^{-8} \sum_{j=1}^n s_{j-1}^2 - 2^{-4}, \end{aligned} \tag{7.2}$$

where in the last step we used the estimate

$$\sum_{j=1}^n s_j^2 - \sum_{j=2}^n s_j s_{j-1} \geq 0,$$

following from the Cauchy-Schwarz inequality, and the estimate  $s_1 s_0 \leq 4$ , following from the assumption (1.10). Furthermore,

$$\int_1^{2^{n+1}} \frac{\theta^2(t)}{t^3} dt = \sum_{j=0}^n \int_{2^j}^{2^{j+1}} \frac{\theta^2(t)}{t^3} dt \leq \sum_{j=0}^n \frac{\theta^2(2^{j+1})}{2^{2j}} = 4 \sum_{j=0}^n s_{j+1}^2. \tag{7.3}$$

Thus, (7.2), (7.3) and (1.11) imply (1.9), i.e.,

$$\sum_{j=1}^{\infty} \left( \frac{d_j - c_j}{|c_j| + 1} \right)^2 = \infty.$$

To complete the proof we apply Remark 2.

□

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